

# Dynamics of Asynchronous Systems

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## Abstract

We introduce partial actions of trace monoids (free partially commutative monoids) over finite sets as a model of asynchronous dynamical system. We characterise Markov measures for these systems by finitely many parameters with suitable normalisation conditions.

The uniform Markov measure attached to an action is constructed. It is characterised by a real number  $t_0$ , the characteristic root of the action, and a function of pairs of states, the Parry cocycle. A new combinatorial inversion formula allows to identify a polynomial of which  $t_0$  is the smallest positive root.

Examples based on simple combinatorial tilings are studied. We introduce the tip-top action, a natural action of a trace monoid on its set of independence cliques. It identifies for certain monoids with the action of flips on the domino tilings of a thin strip.

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## 1—Introduction

The emerging of network systems over the past decades has revealed the need for taking into account a particular feature of certain dynamical systems, namely their asynchronous behaviour. Concurrency theory from Computer Science has studied various models involving asynchrony. But few has been done for giving a mathematical ground to a notion of dynamical system involving an intrinsic notion of *asynchronous actions*. This paper is an attempt to establish such a ground.

The notion of asynchronous systems has applications outside the field of Computer Science. Network systems, in a broad sense, can be seen everywhere. In a chemical solution with several products and several enabled reactions for instance, different chemical reactions can occur concurrently: there are potential applications in natural and physical sciences as well.

A convenient starting point for expressing the asynchrony, or parallelism, of two “actions”, say  $a$  and  $b$ —whatever their exact nature is—, is the commutativity relation  $ab = ba$ . It renders the essential feature that the order of execution of these two actions is irrelevant. These considerations naturally lead—as noticed by several authors [12, 11]—to considering a presented monoid  $\mathcal{M}$  of the form:

$$\mathcal{M} = \langle \Sigma \mid ab = ba \rangle,$$

for  $(a, b)$  ranging over a fixed symmetric and irreflexive relation on the set  $\Sigma$  of generators.

In the literature, such monoids are called free partially commutative monoids, heaps monoids or *trace monoids* [9, 30, 11, 29, 18]. They have a nice interpretation in Combinatorics through Viennot’s heaps of pieces theory. Because of their applications

in both fields, trace monoids stand at an intersection point between Computer Science and Combinatorics.

In most relevant applications, the notion of *state* of a system is essential. A natural structure for an asynchronous system is thus a pair  $(X, \mathcal{M})$ , where  $X$  is a finite set of states, and  $\mathcal{M}$  is a trace monoid, together with a right monoid action  $\varphi : X \times \mathcal{M} \rightarrow X$  of  $\mathcal{M}$  over  $X$ , denoted by  $\varphi(\alpha, x) = \alpha \cdot x$ . If 1 denotes the unit element of the monoid, the monoid action obeys the two following axioms:

$$\alpha \cdot 1 = \alpha, \quad \alpha \cdot (x \cdot y) = (\alpha \cdot x) \cdot y.$$

It is worth observing that acceptor graphs are a particular instance of the above. Indeed, let  $A$  be a finite set, and let  $M$  be an incidence matrix over  $A$ . Hence  $M = (M_{i,j})_{(i,j) \in A \times A}$  satisfies  $M_{i,j} \in \{0, 1\}$ , and the system can go from state  $i$  to state  $j$  in one step if and only if  $M_{i,j} = 1$ . Let:

$$\Sigma = \{(i, j) \in A \times A : M_{i,j} = 1\},$$

thought of as the set of admissible elementary actions, and let  $\Sigma^*$  be the free monoid generated by the admissible actions.

Consider a symbol  $\perp \notin A$ , and put  $A' = A \cup \{\perp\}$ . Then the dynamics of the system corresponds to the unique right action  $A' \times \Sigma^* \rightarrow A'$  such that:

$$\forall i \in A' \quad \forall (j, k) \in \Sigma \quad i \cdot (j, k) = \begin{cases} \perp, & \text{if } i \neq j \text{ or if } i = \perp \\ k, & \text{if } i = j \end{cases}$$

The acceptor graph defines thus a “partial action” of the free monoid  $\Sigma^*$  over the set of states  $A$ . The action is “partial” in the sense that not all actions are always enabled, depending on the current state of the system. But, up to adding a distinguished state  $\perp$ , partial actions are actually a particular instance of a standard action  $A' \times \Sigma^* \rightarrow A'$ , with the additional feature that the trajectories to be considered are those that avoid to ever hit the distinguished state  $\perp$ . Hence these correspond to partial actions without asynchrony; and at a theoretical level, partial actions can be treated as normal actions of a free monoid over a set of states.

The generalisation of “partial actions” to asynchronous systems will be central in this work. Formally, an asynchronous system consists of a pair  $(X, \mathcal{M})$ , where  $X$  is a finite set of states,  $\mathcal{M}$  is a trace monoid, together with a right action  $X \times \mathcal{M} \rightarrow X$ . The set  $X$  is equipped with a distinguished state  $\perp$ ; “trajectories” to consider will be those that avoid to ever hit the distinguished state  $\perp$ .

Dynamics is introduced by means of probability. We define a *Markov measure* over an asynchronous system  $(X, \mathcal{M})$ , as a family  $\mathbb{P} = (\mathbb{P}_\alpha)_{\alpha \in X}$  of probability measures indexed by the set of states, and obeying the following chain rule, without giving the full definition of all terms for now:

$$\forall \alpha \in X \quad \forall x, y \in \mathcal{M} \quad \mathbb{P}_\alpha(x \cdot y) = \mathbb{P}_\alpha(x) \mathbb{P}_{\alpha \cdot x}(y)$$

If  $X$  is a singleton, or if  $\mathbb{P}_\alpha$  is independent of  $\alpha$ , then the property becomes  $\mathbb{P}(x \cdot y) = \mathbb{P}(x) \mathbb{P}(y)$ , which corresponds to Bernoulli measures—or memoryless measures. Bernoulli measures on trace monoids have been the topic of a previous work co-authored with J. Mairesse [4]. The present work covers the generalisation to the Markovian case.

We characterise all Markov measures associated with an asynchronous system  $(X, \mathcal{M})$  through a finite family of probabilistic parameters with certain normalisation conditions. These normalisation conditions are polynomial in the parameters; they are expressed by means of a fundamental combinatorial device, the Möbius transform, particularised for the framework of trace monoids.

Markov measures are our main object of study in this paper. Specially interesting is the case where the asynchronous action is only partially defined. We then require that  $\mathbb{P}_\alpha(x) = 0$  whenever  $x$  is an action not enabled at state  $\alpha$ . Their existence is not obvious. For accepto graphs—hence, without the asynchrony feature—, the Parry construction [22, 16, 20] demonstrates the existence of a “uniform” measure, Markovian, and charging with positive probability every admissible finite trajectory if the accepto graph is irreducible. The generalisation of this measure to asynchronous systems is the second contribution of this paper.

We show that the uniform measure of an asynchronous system is a Markov measure  $\nu = (\nu_\alpha)_{\alpha \in X}$  in our sense. It has the following form, for  $x$  enabled at state  $\alpha$ :

$$\nu_\alpha(x) = t_0^{|x|} \Gamma(\alpha, \alpha \cdot x) \quad |x| = \text{length of } x$$

Here,  $t_0$  is a real number lying in  $(0, 1)$  and is called the *characteristic root* of the action; and  $\Gamma(\cdot, \cdot) : X \times X \rightarrow \mathbb{R}$  is a positive function, called the *Parry cocycle*, with the following property:  $\Gamma(\alpha, \beta) \Gamma(\beta, \gamma) = \Gamma(\alpha, \gamma)$ . This holds under an irreducibility assumption on the action.

Determining  $t_0$  and the Parry cocycle is challenging when facing concrete examples. We introduce a simple combinatorial example of asynchronous action, related to tiling models. We illustrate our theoretical results by solving the problem of determining both the characteristic root and the Parry cocycle in several different ways for this example.

The growth series  $\sum_{x \in \mathcal{M}} t^{|x|}$  of a trace monoid  $\mathcal{M}$  is a well studied object. It is given by the inverse of the Möbius polynomial of the monoid (also called the independence polynomial in a graph theoretic context [19]). This directly relates the radius of convergence of the series with the root of smallest modulus of the Möbius polynomial. One method for obtaining this inversion formula is based on the theory of formal series over partially commutative variables [9]. In our context, where not only the monoid but also its action over a set is involved, we have discovered that a slight generalisation of this theory allows to obtain similar results. It is thus another contribution of this paper to introduce *formal fibred series over non commutative variables*, and to show their application for determining the radius of convergence of growth series of the form:

$$Z_\alpha(t) = \sum_{\substack{x \in \mathcal{M} : \\ x \text{ enabled at } \alpha}} t^{|x|}$$

The characteristic root  $t_0$  corresponds to the common radius of convergence of the above power series, for  $\alpha$  ranging over the set of states. We obtain a generalisation of the Möbius polynomial of a trace monoid, of which  $t_0$  is the smallest root. This amounts to encoding not only the combinatorics of the monoid, but also of the action, into a certain polynomial with integer coefficients.

*Organisation* — In order to keep the paper self-contained, we have gathered in Section 2 some background on trace monoids. We have also included a summary on Bernoulli measures for trace monoids; although we shall not use the related results directly, their comparison with subsequent results on Markov measures will probably be useful to the reader.

Section 3 introduces the notions of total and partial actions of trace monoids. We also introduce the example of the tip-top action of a trace monoid over its set of cliques. We particularise it for the trace monoid generated by the flips of the domino tilings of a strip, producing the Rabati tiling example.

Section 4 is devoted to a general study of Markov measures for actions of trace monoids. We introduce the notion of fibred valuations and of Möbius fibred valuations. We show that Markov measures correspond to Möbius fibred valuations.

Markov measures are given a realisation through the Markov chain of states-and-cliques.

In Section 5, we particularise our study to the case of partial actions. The goal is to show the existence of a Markov measure with same support as the action. To this aim, we construct the uniform measure related to the action, and we show that it is Markovian. In the course of this construction, we introduce the characteristic root of the action and its Parry cocycle. We give an inversion formula which allows to interpret the characteristic root of the action as a particular root of a polynomial attached to the action.

## 2—Preliminaries

In this section we collect the needed facts concerning trace monoids and the construction of associated Bernoulli measures. Each time a new notation is introduced, it is understood that we shall later refer to the designated object through this notation.

Bibliographical references are gathered in § 2.5.

### 2.1 — Trace monoids

**2.1.1 Independence relation** — An alphabet  $\Sigma$  is a finite non empty set, and we will always consider that  $|\Sigma| \geq 2$ . Elements of  $\Sigma$  are called letters or pieces. An independence relation  $I$  is a binary irreflexive and symmetric relation on  $\Sigma$ .

**2.1.2 Trace monoid** — Let  $\mathcal{R}_I$  be the smallest congruence on the free monoid  $\Sigma^*$  containing all pairs  $(ab, ba)$  for  $(a, b)$  ranging over  $I$ . The trace monoid  $\mathcal{M} = \mathcal{M}(\Sigma, I)$  is the quotient monoid  $\mathcal{M} = \Sigma^* / \mathcal{R}_I$ .

**2.1.3 Traces** — Elements of  $\mathcal{M}$  are called traces. Letters of  $\Sigma$  are identified with their images in  $\mathcal{M}$  through the canonical morphisms  $\Sigma \rightarrow \Sigma^* \rightarrow \mathcal{M}$ . Concatenation in  $\mathcal{M}$  is denoted with the dot “.”. The unit element is denoted “1” and is called the empty trace.

**2.1.4 Immediate equivalence** — Let  $\mathcal{R}_I$  be the binary relation on  $\Sigma^*$  containing all pairs of words of the form  $(xaby, xbay)$ , for  $x, y \in \Sigma^*$  and  $(a, b) \in I$ . Then  $\mathcal{R}_I$  is the reflexive and transitive closure of  $\mathcal{R}_I$ .

**2.1.5 Length of traces** — Since two  $\mathcal{R}_I$ -related words have the same length, and since  $\mathcal{R}_I$  is the transitive and reflexive closure of  $\mathcal{R}_I$ , all representative words of a given trace have the same length. This defines thus a mapping  $|\cdot| : \mathcal{M} \rightarrow \mathbb{N}$ , and  $|x|$  is called the length of trace  $x$ . The length is additive:  $|x \cdot y| = |x| + |y|$ . The empty trace is the unique trace of length 0, and letters are the only traces of length 1.

**2.1.6 Divisibility relation** — The left divisibility relation in  $\mathcal{M}$  is denoted “ $\leq$ ”:  $x \leq y \iff \exists z \in \mathcal{M} \ y = x \cdot z$ . As in any monoid, it is reflexive and transitive. From the additivity of length, it is also easily seen to be anti-symmetric, hence  $(\mathcal{M}, \leq)$  is a partially ordered set.

**2.1.7 Compatible traces** — Two traces  $x, y \in \mathcal{M}$  have a least upper bound  $x \vee y$  with respect to  $\leq$  if and only if there exists a trace  $z \in \mathcal{M}$  such that  $x \leq z$  and  $y \leq z$ . In this case,  $x$  and  $y$  are said to be compatible.

**2.1.8 Cliques** — A clique of  $\mathcal{M}$  is a trace of the form  $x = a_1 \cdot \dots \cdot a_i$ , where  $a_i$  are pairwise distinct letters such that  $i \neq j \implies (a_i, a_j) \in I$ . Consider the pair  $(\Sigma, I)$  as a graph. Then cliques are in bijection with those subgraphs of  $(\Sigma, I)$  which are complete graphs—these are indeed called cliques in a graph theoretic context. Restricted to cliques, the ordering relation  $\leq$  corresponds to the inclusion of subsets, when seeing cliques as subsets of  $\Sigma$ . We denote by  $\mathcal{C}$  the set of cliques, and by  $\mathcal{C} = \mathcal{C} \setminus \{1\}$  the set of non empty cliques.

**2.1.9 Parallelism of cliques** — Two cliques  $c, c' \in \mathcal{C}$  are said to be parallel, denoted  $c \parallel c'$ , whenever  $c \cdot c' \in \mathcal{C}$ . When seeing cliques as subsets of  $\Sigma$ , this is equivalent to:

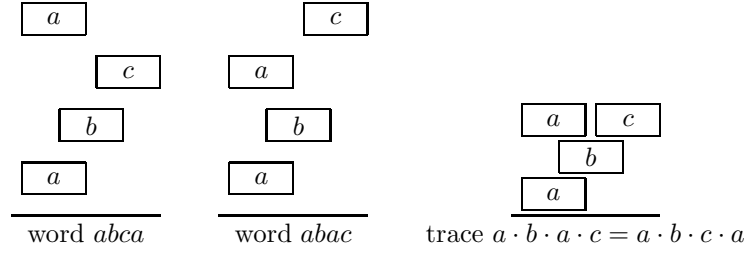


Figure 1: Two congruent words and the resulting heap (trace) for  $\mathcal{M} = \langle a, b, c \mid a \cdot c = c \cdot a \rangle$ . Pieces  $a$  and  $c$  fall in a parallel way, but not  $a$  and  $b$  and neither  $c$  and  $b$ .

$c \times c' \subseteq I$ . In particular, and since single letters are cliques by themselves, we have  $a \parallel a' \iff (a, a') \in I$  if  $a, a' \in \Sigma$ .

**2.1.10 Cartier-Foata relation** — Two cliques  $c, c' \in \mathcal{C}$  are Cartier-Foata compatible, denoted by  $c \rightarrow c'$ , if for every letter  $b \in c'$  there exists a letter  $a \in c$  such that  $(a, b) \notin I$ .

**2.1.11 Cartier-Foata normal form. Height of traces** — For every non empty trace  $x \in \mathcal{M} \setminus \{1\}$ , there exists a unique integer  $n \geq 1$  and a unique sequence  $(c_1, \dots, c_n)$  of non empty cliques such that: (1)  $c_i \rightarrow c_{i+1}$  holds for all  $i \in \{1, \dots, n-1\}$ ; and (2)  $x = c_1 \cdot \dots \cdot c_n$ . The sequence  $(c_1, \dots, c_n)$  is called Cartier-Foata decomposition (or normal form) of  $x$ .

The integer  $n$  is called the height of  $x$ , denoted by  $n = \tau(x)$ . By convention, we put  $\tau(1) = 0$ .

**2.1.12 Heaps of pieces** — Heaps of pieces provide a visually intuitive representation of traces. Picture each letter as a piece, or domino, falling from top to bottom until it reaches the ground or a previously placed piece. Two pieces  $a, b \in \Sigma$  are bound to fall in a parallel way with respect to each other if and only if  $(a, b) \in I$ , which renders the commutativity relation  $a \cdot b = b \cdot a$ . Then it is part of Viennot's theory that heaps of pieces thus obtained are in bijection with traces. We illustrate the heap representation in Figure 1.

**2.1.13 Layers of heaps and Cartier-Foata normal form** — In the heap of pieces representation of traces, the cliques that appear in the Cartier-Foata decomposition correspond to the successive horizontal layers, from bottom to top, that compose the heap. For the trace depicted in Figure 1, the Cartier-Foata decomposition is  $a \rightarrow b \rightarrow a \cdot c$ .

## 2.2 — Growth series and Möbius inversion formulas

**2.2.1 Growth series and Möbius polynomial** — The growth series of the monoid  $\mathcal{M}$  is the power series  $H(t)$  defined by:

$$H(t) = \sum_{x \in \mathcal{M}} t^{|x|}.$$

The Möbius polynomial  $\mu_{\mathcal{M}}(t)$  of  $\mathcal{M}$  is defined by:

$$\mu_{\mathcal{M}}(t) = \sum_{\gamma \in \mathcal{C}} (-1)^{|\gamma|} t^{|\gamma|}.$$

For instance, for  $\mathcal{M} = \langle a, b, c \mid a \cdot c = c \cdot a \rangle$ , illustrated in Figure 1, one has  $\mathcal{C} = \{1, a, b, c, a \cdot c\}$  and thus  $\mu_{\mathcal{M}}(t) = 1 - 3t + t^2$ .

**2.2.2 First Möbius inversion formula** — As a formal series, the growth series  $H(t)$  is rational, inverse of the Möbius polynomial:

$$H(t) = \frac{1}{\mu_{\mathcal{M}}(t)}. \quad (1)$$

**2.2.3 Smallest root of the Möbius polynomial** — The Möbius polynomial has a unique root of smallest modulus, which is real and lies in  $(0, 1)$ . This root coincides with the unique dominant singularity of  $H(t)$ .

**2.2.4 Möbius transform** — Let  $f : \mathcal{C} \rightarrow \mathbb{R}$  be a function (it could actually take its values in any commutative monoid). The Möbius transform of  $f$  is the function  $h : \mathcal{C} \rightarrow \mathbb{R}$  defined by:

$$\forall c \in \mathcal{C} \quad h(c) = \sum_{c' \in \mathcal{C} : c' \geq c} (-1)^{|c'| - |c|} f(c'). \quad (2)$$

**2.2.5 Graded Möbius transform** — If  $f : \mathcal{M} \rightarrow \mathbb{R}$  is defined on  $\mathcal{M}$ , and not only on  $\mathcal{C}$ , we extend its Möbius transform as follows. It is defined as in (2) on  $\mathcal{C}$ . For  $\tau(x) \geq 2$ , let  $x = c_1 \rightarrow \dots \rightarrow c_n$  be the Cartier-Foata decomposition of  $x$ , and let  $y = c_1 \cdot \dots \cdot c_{n-1}$ . Then we define:

$$h(x) = \sum_{c \in \mathcal{C} : c \geq c_n} (-1)^{|c| - |c_n|} f(y \cdot c).$$

The function  $h : \mathcal{M} \rightarrow \mathbb{R}$  is called the graded Möbius transform of  $f$ .

**2.2.6 Second Möbius inversion formula** — Define  $\mathcal{M}(1) = \mathcal{C}$  and, for each non empty trace  $x \in \mathcal{M}$ , put  $\mathcal{M}(x) = \{y \in \mathcal{M} : \tau(y) = \tau(x) \wedge x \leq y\}$ . Then, for any function  $f : \mathcal{M} \rightarrow \mathbb{R}$ , with graded Möbius transform  $h : \mathcal{M} \rightarrow \mathbb{R}$ , holds:

$$\forall x \in \mathcal{M} \quad f(x) = \sum_{y \in \mathcal{M}(x)} h(y). \quad (3)$$

Conversely, if  $f, h : \mathcal{M} \rightarrow \mathbb{R}$  are two functions such that (3) holds, then  $h$  is the graded Möbius transform of  $f$ .

## 2.3 — Boundary and compactification

**2.3.1 Generalised traces** — There exists a canonical partial order  $(\overline{\mathcal{M}}, \leq)$ , which elements are called generalised traces, and with the following properties:

1. Every non decreasing sequence in  $\overline{\mathcal{M}}$  has a least upper bound.
2. There is a canonical embedding of partial orders  $\iota : \mathcal{M} \rightarrow \overline{\mathcal{M}}$ , hence we identify  $\mathcal{M}$  as a subset of  $\overline{\mathcal{M}}$ .
3. Every element of  $\overline{\mathcal{M}}$  is the least upper bound  $\bigvee \{x_n : n \geq 1\}$  of a non decreasing sequence  $(x_n)_{n \geq 1}$  with  $x_n \in \mathcal{M}$  for all  $n \geq 1$ .

**2.3.2 Infinite traces** — Let  $\partial\mathcal{M} = \overline{\mathcal{M}} \setminus \mathcal{M}$ . Elements of  $\partial\mathcal{M}$  are called infinite traces, and  $\partial\mathcal{M}$  is the boundary at infinity, or simply the boundary of  $\mathcal{M}$ . For every  $\xi \in \partial\mathcal{M}$ , there exists a unique infinite sequence  $(c_i)_{i \geq 1}$  of non empty cliques such that holds:

$$\xi = \bigvee_{i \geq 1} (c_1 \cdot \dots \cdot c_i), \quad \forall i \geq 1 \quad c_i \rightarrow c_{i+1}.$$

The sequence  $(c_i)_{i \geq 1}$  extends to infinite traces the Cartier-Foata decomposition of traces (§ 2.1.11).

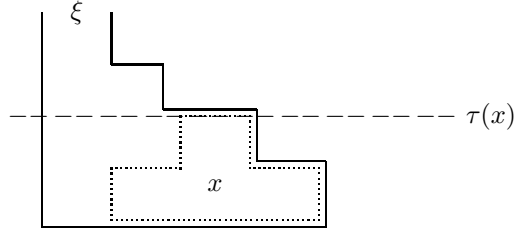


Figure 2: Illustrating the ordering relation  $x \leq \xi$  between a trace  $x$ , symbolised by the dotted line  $\cdots$ , and an infinite trace  $\xi$ , symbolised by the solid line  $\text{—}$ . The height  $\tau(x)$  of  $x$  is depicted.

**2.3.3 Ordering on generalised traces** — For each integer  $n \geq 0$ , the  $n$ -topping is the mapping  $\kappa_n : \overline{\mathcal{M}} \rightarrow \mathcal{M}$  defined by  $\kappa_n(\xi) = c_1 \cdot \dots \cdot c_n$ , where  $(c_i)_{i \geq 1}$  is the extended Cartier-Foata decomposition of  $\xi$ , maybe with  $c_i = 1$  whenever  $\xi \in \mathcal{M}$  and  $i > \tau(\xi)$ .

The ordering on traces is defined by the left divisibility relation (§ 2.1.6). Its extension on  $\overline{\mathcal{M}}$  can be characterised as follows: for all  $\xi, \xi' \in \overline{\mathcal{M}}$ ,  $\xi \leq \xi'$  holds if and only if  $\kappa_n(\xi) \leq \kappa_n(\xi')$  for all integers  $n \geq 0$ . For  $x \in \mathcal{M}$  and  $\xi \in \partial\mathcal{M}$ , the relation  $x \leq \xi$  reduces to this:

$$x \leq \xi \iff x \leq \kappa_{\tau(x)}(\xi).$$

The visual intuition of this result is illustrated on Figure 2.

**2.3.4 Elementary cylinders** — For  $x \in \mathcal{M}$ , the elementary cylinder of base  $x$  is the subset of  $\partial\mathcal{M}$  denoted by  $\uparrow x$  and defined by:

$$\uparrow x = \{\xi \in \partial\mathcal{M} : x \leq \xi\}.$$

The existence of least upper bounds for compatible traces (§ 2.1.7) implies that elementary cylinders intersect as follows:

$$\uparrow x \cap \uparrow y = \begin{cases} \emptyset, & \text{if } x \text{ and } y \text{ are not compatible,} \\ \uparrow (x \vee y), & \text{if } x \text{ and } y \text{ are compatible.} \end{cases}$$

**2.3.5 Topology on  $\partial\mathcal{M}$  and on  $\overline{\mathcal{M}}$ . Compactness** — For each  $x \in \mathcal{M}$ , let the full elementary cylinder  $\uparrow\uparrow x$  be defined by:

$$\uparrow\uparrow x = \{\xi \in \overline{\mathcal{M}} : x \leq \xi\}. \quad (4)$$

The topology we consider on  $\overline{\mathcal{M}}$  corresponds to the Lawson topology in Domain theory (see [13]). It is the join of the topologies  $\mathcal{S}$  and  $\mathcal{L}$ , i.e., the smallest topology containing both  $\mathcal{S}$  and  $\mathcal{L}$ . The topologies  $\mathcal{S}$  (Scott topology) and  $\mathcal{L}$  (lower topology) are the topologies generated by the following subsets:

$$\begin{aligned} \mathcal{S} : & \uparrow\uparrow x, \text{ for } x \in \mathcal{M} \\ \mathcal{L} : & \overline{\mathcal{M}} \setminus (\uparrow\uparrow x), \text{ for } x \in \overline{\mathcal{M}} \end{aligned}$$

The boundary  $\partial\mathcal{M}$  is equipped with the restriction of this topology. For our concern, we will only need the following facts regarding these topologies:

1. For each trace  $x \in \mathcal{M}$ , the singleton  $\{x\}$  is both open and closed in  $\overline{\mathcal{M}}$ .
2. For each trace  $x \in \mathcal{M}$ ,  $\uparrow x$  is both open and closed in  $\partial\mathcal{M}$ , and  $\uparrow\uparrow x$  is both open and closed in  $\overline{\mathcal{M}}$ .
3. The space  $\overline{\mathcal{M}}$  is metrisable and compact; the subset  $\partial\mathcal{M}$  is closed in  $\overline{\mathcal{M}}$ .

## 2.4 — Bernoulli and uniform measures on the boundary

**2.4.1  $\sigma$ -algebras and  $\pi$ -systems on  $\overline{\mathcal{M}}$  and on  $\partial\mathcal{M}$**  — We equip  $\overline{\mathcal{M}}$  and  $\partial\mathcal{M}$  with their respective Borel  $\sigma$ -algebras,  $\overline{\mathfrak{F}}$  and  $\mathfrak{F}$ . The  $\sigma$ -algebra on  $\overline{\mathcal{M}}$  is generated by the collection of full elementary cylinders  $\uparrow x$ , for  $x$  ranging over  $\mathcal{M}$ , defined in § 2.3.5.

By the intersection property of cylinders (§ 2.3.4, see also § 2.1.7), we observe that both collections:

$$\{\emptyset\} \cup \{\uparrow x : x \in \mathcal{M}\}, \quad \{\emptyset\} \cup \{\uparrow x : x \in \mathcal{M}\},$$

are  $\pi$ -systems (i.e.: stable under finite intersections) generating  $\mathfrak{F}$  and  $\overline{\mathfrak{F}}$ , respectively.

**2.4.2 Valuations** — A valuation on  $\mathcal{M}$  is a function  $f : \mathcal{M} \rightarrow \mathbb{R}$  such that  $f(1) = 1$  and  $f(x \cdot y) = f(x)f(y)$  for all  $x, y \in \mathcal{M}$ . Valuations are in bijection with families or real numbers  $(\lambda_a)_{a \in \Sigma}$ . The correspondence assigns to each valuation  $f$  the collection  $(f(a))_{a \in \Sigma}$ . The valuation is uniform if it is constant on  $\Sigma$ . It corresponds to:

$$\forall x \in \mathcal{M} \quad f(x) = p^{|x|}, \quad \forall a \in \Sigma \quad p = f(a).$$

**2.4.3 Bernoulli measures** — A Bernoulli measure is a probability measure on  $(\partial\mathcal{M}, \mathfrak{F})$  such that:

$$\forall x, y \in \mathcal{M} \quad \mathbb{P}(\uparrow(x \cdot y)) = \mathbb{P}(\uparrow x)\mathbb{P}(\uparrow y).$$

If  $\mathbb{P}$  is a Bernoulli measure, the function  $f : x \in \mathcal{M} \mapsto \mathbb{P}(\uparrow x)$  is the valuation associated with  $\mathbb{P}$ .

**2.4.4 Möbius valuations** — Let  $f : \mathcal{M} \rightarrow \mathbb{R}$  be a valuation, and let  $h : \mathcal{M} \rightarrow \mathbb{R}$  be the Möbius transform of  $f$  (§ 2.2.4). By definition,  $f$  is a Möbius valuation if:

$$h(1) = 0, \quad \forall c \in \mathfrak{C} \quad h(c) \geq 0.$$

**2.4.5 Characterisation of Bernoulli measures** — If  $\mathbb{P}$  is a Bernoulli measure, then its associated valuation is Möbius. Conversely, let  $f : \mathcal{M} \rightarrow \mathbb{R}$  be a Möbius valuation. Then there exists a unique Bernoulli measure  $\mathbb{P}$  such that  $\mathbb{P}(\uparrow x) = f(x)$  for all  $x \in \mathcal{M}$ .

**2.4.6 Random decomposition of infinite traces** — For each integer  $i \geq 1$ , let  $C_i : \partial\mathcal{M} \rightarrow \mathfrak{C}$  be the mapping which assigns to an infinite trace  $\xi$  the  $i^{\text{th}}$  clique in its extended Cartier-Foata decomposition. Let also:

$$Y_i = C_1 \cdot \dots \cdot C_i,$$

be defined for all non negative integers, with  $Y_0 = 1$  for  $i = 0$ . Then  $(C_i)_{i \geq 1}$  and  $(Y_i)_{i \geq 0}$  are two sequences of measurable mappings.

**2.4.7 Markov chain of cliques** — Under a Bernoulli measure  $\mathbb{P}$ , the sequence  $(C_i)_{i \geq 1}$  is a Markov chain. The law of  $C_1$  is given by the restriction  $h|_{\mathfrak{C}}$ , where  $h : \mathfrak{C} \rightarrow \mathbb{R}$  is the Möbius transform of the associated valuation. In case the valuation  $f(\cdot) = \mathbb{P}(\uparrow \cdot)$  is positive on  $\mathcal{M}$ , then the transition matrix  $P = (P_{c,c'})_{(c,c') \in \mathfrak{C} \times \mathfrak{C}}$  of the Markov chain  $(C_i)_{i \geq 1}$  is given by:

$$P_{c,c'} = \mathbf{1}_{\{c \rightarrow c'\}} \frac{h(c')}{g(c)}, \quad g(c) = \sum_{c' \in \mathfrak{C} : c \rightarrow c'} h(c').$$

**2.4.8 Uniform measure** — There exists a unique Bernoulli measure  $\nu$  such that the associated valuation is uniform. It is given by:

$$\forall x \in \mathcal{M} \quad \nu(\uparrow x) = p_0^{|x|},$$

where  $p_0$  is the unique root of smallest modulus of the Möbius polynomial  $\mu_{\mathcal{M}}$  (§ 2.2.1).



**2.4.9 Complement on elementary cylinders** — It is useful to relate elementary cylinders (§ 2.3.4) and  $Y_i$ -measurable subsets of  $\partial\mathcal{M}$ , where  $Y_i$  has been defined in § 2.4.6. Let  $x$  be a non empty trace of height  $n = \tau(x)$ . Then the elementary cylinder  $\uparrow x$  decomposes as the following disjoint union:

$$\uparrow x = \bigcup_{z \in \mathcal{M}(x)} \{Y_n = z\}, \quad (5)$$

where  $\mathcal{M}(x) = \{z \in \mathcal{M} : \tau(z) = \tau(x) \wedge x \leq z\}$  has been introduced in § 2.2.6.

## 2.5 — Bibliographical references

The basics of trace monoids are covered, under different points of view, in [9, 30, 11, 12]. A famous application of trace monoids to the combinatorics of directed animals is found in [8].

The first Möbius inversion formula is established in [9] and in [30]. The properties of the Möbius polynomial of a trace monoid are studied in [14, 18, 10]. The second inversion formula for graded Möbius transform is established in [1]. The simple Möbius transform is a particular instance of the notion of Möbius transform [24, 28].

The space of generalised traces is introduced in [4]. It is a particular instance of the completion of certain presented monoids studied in [2], and is also a particular instance of the completion of partial orders [15]. The topological properties are rather standard; they can, for instance, be established within the framework of [3].

Bernoulli and uniform measures for trace monoids are introduced in [4], in which only irreducible trace monoids were considered. Relaxing this assumption does not present major difficulties, as shown in [5].

## 3—Actions of trace monoids

### 3.1 — Total actions

A right action of a monoid  $\mathcal{M}$  with unit 1 over a set  $X$  is defined by a mapping  $\varphi : X \times \mathcal{M} \rightarrow X$ , usually denoted by  $\varphi(\alpha, x) = \alpha \cdot x$ , and satisfying the following properties: (1)  $\forall \alpha \in X \quad \alpha \cdot 1 = \alpha$ ; (2)  $\forall \alpha \in X \quad \forall x, y \in \mathcal{M} \quad (\alpha \cdot x) \cdot y = \alpha \cdot (x \cdot y)$ .

In order to emphasise the distinction with partial actions to be defined later, we introduce total actions as follows.

• **Definition 3.1**—A *total action* is a triple  $(X, \mathcal{M}, \varphi)$ , where  $X$  is a finite and non empty set,  $\mathcal{M}$  is a trace monoid, and  $\varphi : X \times \mathcal{M} \rightarrow X$  defines an action of  $\mathcal{M}$  over  $X$ . We usually denote the action by  $\varphi(\alpha, x) = \alpha \cdot x$ . In this context, elements of  $X$  are called *states*, elements of  $\Sigma$  are called *elementary actions*, and elements of  $\mathcal{M}$  are called *actions*.

Assume given a total action of a trace monoid  $\mathcal{M} = \mathcal{M}(\Sigma, I)$  over  $X$ . Let  $\alpha \in X$ , and let  $a, a' \in \Sigma$  with  $a \parallel a'$  (see § 2.1.9). Then  $a \cdot a' = a' \cdot a$  and therefore:  $(\alpha \cdot a) \cdot a' = (\alpha \cdot a') \cdot a$ . The following proposition states a converse to this observation.

• **Proposition 3.2**—Let  $\mathcal{M} = \mathcal{M}(\Sigma, I)$  be a trace monoid, let  $X$  be a finite set, and let  $(\varphi_a)_{a \in \Sigma}$  be a family of mappings  $\varphi_a : X \rightarrow X$  indexed by  $\Sigma$ . Assume that holds:

$$\forall a, a' \in \Sigma \quad a \parallel a' \implies \varphi_a \circ \varphi_{a'} = \varphi_{a'} \circ \varphi_a.$$

Then there exists a unique total action  $\varphi : X \times \mathcal{M} \rightarrow X$  such that  $\varphi(\cdot, a) = \varphi_a$  for all  $a \in \Sigma$ .

*Proof.* Immediate by the universal property of quotient monoids.  $\square$

## 3.2 — Partial actions

**3.2.1 Definition** — For several applications, it is desirable that some actions may not be always “enabled”, depending on the current state. It is this very feature that we aim at in introducing the following definition.

• **Definition 3.3**—A *partial action* of a trace monoid  $\mathcal{M} = \mathcal{M}(\Sigma, I)$  over a finite set  $X$  is defined by:

1. A collection of subsets  $(\Sigma(\alpha))_{\alpha \in X}$  with  $\Sigma(\alpha) \subseteq \Sigma$  and  $\Sigma(\alpha) \neq \emptyset$ , where  $\Sigma(\alpha)$  corresponds to the set of enabled elementary actions at state  $\alpha$ .
2. A collection  $(\psi_\alpha)_{\alpha \in X}$  of mappings, with  $\psi_\alpha : \Sigma(\alpha) \rightarrow X$  denoted  $\psi_\alpha(a) = \alpha \cdot a$  whenever  $a \in \Sigma(\alpha)$ , such that for any  $a, b \in \Sigma$ , if  $a \in \Sigma(\alpha)$  and if  $a \parallel b$ , then:
  - (i) Either  $b \in \Sigma(\alpha)$ ; and in this case  $a \in \Sigma(\alpha \cdot b)$  and  $b \in \Sigma(\alpha \cdot a)$  and  $(\alpha \cdot a) \cdot b = (\alpha \cdot b) \cdot a$ ;
  - (ii) Or  $b \notin \Sigma(\alpha)$ ; and in this case  $b \notin \Sigma(\alpha \cdot a)$ .

The intuition explaining the two properties of point 2 in the above definition is that an action  $b \in \Sigma$  which is not enabled at some state  $\alpha \in X$  suffers from some sort of lock on it. The idea is that a parallel action  $a \parallel b$  cannot unlock  $b$ ; and neither can it lock  $b$  if  $b$  was enabled. See a concrete example in § 3.3.3 below.

**3.2.2 Total action associated to a partial action** — We now introduce a construction which associates a canonical total action to each partial action. Assume given a partial action with the same notations as in Definition 3.3. Let  $\perp$  be an element not in  $X$ , and let  $X' = X \cup \{\perp\}$ . For each  $\alpha \in X$ , extend  $\psi_\alpha : \Sigma(\alpha) \rightarrow X$  to a mapping  $\psi_\alpha : \Sigma \rightarrow X$  by putting  $\psi_\alpha(a) = \perp$  if  $a \notin \Sigma(\alpha)$ , and put  $\psi_\perp(\cdot) = \perp$ .

• **Proposition 3.4**—There exists a unique total action of  $\mathcal{M}$  over  $X'$  such that  $\alpha \cdot a = \psi_\alpha(a)$  for all  $\alpha \in X'$  and for all  $a \in \Sigma$ .

*Proof.* For each  $a \in \Sigma$ , let  $\varphi_a : X' \rightarrow X'$  be defined by  $\varphi_a(\alpha) = \psi_\alpha(a)$ . We verify that  $(\varphi_a)_{a \in \Sigma}$  satisfies the condition of Proposition 3.2. For this, let  $a, b \in \Sigma$  with  $a \parallel b$ , and let  $\alpha \in X'$ . Clearly, for  $\alpha = \perp$ , one has  $\varphi_a \circ \varphi_b(\perp) = \perp = \varphi_b \circ \varphi_a(\perp)$ . And for  $\alpha \in X$ , we discuss different cases.

Assume that neither  $a \in \Sigma(\alpha)$  nor  $b \in \Sigma(\alpha)$  hold. Then:  $\varphi_a(\varphi_b(\alpha)) = \varphi_a(\perp) = \perp$  and  $\varphi_b(\varphi_a(\alpha)) = \varphi_b(\perp) = \perp$ , hence  $\varphi_a \circ \varphi_b(\alpha) = \varphi_b \circ \varphi_a(\alpha)$ .

Assume that both  $a \in \Sigma(\alpha)$  and  $b \in \Sigma(\alpha)$  hold. Then, by point (2i) of Definition 3.3, the equality  $\varphi_a \circ \varphi_b(\alpha) = \varphi_b \circ \varphi_a(\alpha)$  holds.

Assume that, of  $a$  and  $b$ , only one is enabled at state  $\alpha$ , say  $a \in \Sigma(\alpha)$  and  $b \notin \Sigma(\alpha)$ . Then on the one hand:  $\varphi_a(\varphi_b(\alpha)) = \varphi_a(\perp) = \perp$ . And on the other hand,  $b \notin \Sigma(\alpha \cdot a)$  by point (2ii) of Definition 3.3, and thus:  $\varphi_b(\varphi_a(\alpha)) = \perp$ , whence the equality  $\varphi_a \circ \varphi_b(\alpha) = \varphi_b \circ \varphi_a(\alpha)$ .

The result follows then from Proposition 3.2.  $\square$

**3.2.3 Terminology** — We speak of a “partial action  $X \times \mathcal{M} \rightarrow X$ ” when considering the total action of  $\mathcal{M}$  over  $X$  resulting from the construction of § 3.2.2 with respect to a partial action of  $\mathcal{M}$  over  $X \setminus \{\perp\}$ , keeping the notation  $\perp$  for the distinguished state.

**3.2.4 Trace language associated with a partial action** — We introduce below a collection of trace languages (a trace language is a subset of a trace monoid) associated with a partial action.

• **Definition 3.5**—Let  $X \times \mathcal{M} \rightarrow X$  be a partial action. For each state  $\alpha \in X$ , we define the subset  $\mathcal{M}_\alpha \subseteq \mathcal{M}$  and the function  $F_\alpha : \mathcal{M} \rightarrow \{0, 1\}$  by:

$$\mathcal{M}_\alpha = \{x \in \mathcal{M} : \alpha \cdot x \neq \perp\}, \quad F_\alpha(x) = \mathbf{1}_{\{\alpha \cdot x \neq \perp\}}.$$

The family  $F = (F_\alpha)_{\alpha \in X}$  is called the support valuation of the partial action.

• **Proposition 3.6**—For each state  $\alpha \in X$  of a partial action  $X \times \mathcal{M} \rightarrow \mathcal{M}$ , the language  $\mathcal{M}_\alpha$  is downward closed:

$$\forall x, y \in \mathcal{M} \quad x \in \mathcal{M}_\alpha \wedge y \leq x \implies y \in \mathcal{M}_\alpha.$$

The support valuation satisfies the following properties:

$$\begin{aligned} \forall \alpha \in X \setminus \{\perp\} \quad F_\alpha(1) &= 1 \\ \forall \alpha \in X \quad \forall x, y \in \mathcal{M} \quad F_\alpha(x \cdot y) &= F_\alpha(x)F_{\alpha \cdot x}(y). \end{aligned}$$

*Proof.* By contraposition, assume that  $x, y \in \mathcal{M}$  are such that  $y \notin \mathcal{M}_\alpha$  and  $y \leq x$ , we have to prove that  $x \notin \mathcal{M}_\alpha$ . Let  $z \in \mathcal{M}$  be such that  $x = y \cdot z$ . Then, by the action property, one has:  $\alpha \cdot x = (\alpha \cdot y) \cdot z = \perp \cdot z$  and an easy induction shows that  $\perp \cdot z = \perp$  for all  $z \in \mathcal{M}$ . Hence  $\alpha \cdot x = \perp$  and thus  $x \notin \mathcal{M}_\alpha$ . The two properties of  $F_\alpha$  follow.  $\square$

### 3.3 — Examples

We introduce examples of actions of trace monoids, some of which will serve as running examples in the remaining of the paper.

**3.3.1 A natural total action** — The heap of pieces interpretation of traces (§ 2.1.12) provides an example of a total action, which we informally describe.

Any heap of pieces, and thus any trace, can be seen as a labelled partial order [30, 17], the labels ranging over the alphabet  $\Sigma$ . As such, it has a number of maximal pieces; let  $M(x)$  be the labelled set of maximal pieces of a heap  $x$ . Let  $X$  be the finite collection of all possible labelled sets  $M(x)$ , for  $x$  ranging over  $\mathcal{M}$ . Then  $\mathcal{M}$  acts on  $X$  by  $M \cdot x = M(y \cdot x)$ , where  $y$  is the flat heap that identifies with  $M$ .

Hence  $M \cdot x$  represents the set of pieces that can be seen, from above, when piling up the heap  $x$  over  $M$ . In case  $\mathcal{M}$  is the free monoid  $\Sigma^*$ , then  $X$  identifies with  $\Sigma$  and  $\alpha \cdot x$  is the last letter of the word  $\alpha x$ . Hence this action generalises in a natural way the Markov chain model, where states and actions coincide.

**3.3.2 Petri nets** — Petri nets, also called place/transition systems and which come in several variants, are a well known model of concurrent systems, introduced in the 1960's and with as various applications as workflow control, formal verification of logical properties of systems, diagnosis and monitoring of discrete events systems, and performance evaluation and queuing network theory on the applied mathematics side [21, 6].

Without entering into the details of the model, let us mention that the executions of a Petri net are given as sequences of transitions. Each transition operates on the marking of the net, which corresponds to the global state of the system. For so-called bounded Petri nets, the set of reachable markings is finite. There is a structural independence relation between transitions [12], which naturally defines a trace monoid generated by the transitions. The action of transitions defines a partial action of this trace monoid on the set of markings of the Petri net. The so-called Petri net language consists then in the trace languages  $\mathcal{M}_\alpha$  of Definition 3.5.

Note that the stochastic models usually attached to Petri nets, as in [6] for instance, randomize the execution sequences of the net, not their equivalence classes modulo commutativity of independent transitions, and differ thus radically from our model.

3.3.3 *The tip-top action* — Let  $\mathcal{M}$  be a trace monoid and let  $X = \mathcal{C}$ , the set of cliques of  $\mathcal{M}$  (§ 2.1.8). We define the tip-top action as the partial action  $\varphi : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{C}$  with:

$$\begin{aligned} \forall \gamma \in \mathcal{C} \quad \Sigma(\gamma) &= \{a \in \Sigma : a \leq \gamma \vee a \parallel \gamma\} \\ \forall \gamma \in \mathcal{C} \quad \forall a \in \Sigma(\gamma) \quad \varphi(\gamma, a) &= \begin{cases} \gamma \setminus \{a\}, & \text{if } a \leq \gamma \\ \gamma \cdot a, & \text{if } a \parallel \gamma \end{cases} \end{aligned}$$

Let us verify that the tip-top action satisfies the condition of Definition 3.3. Let  $\gamma \in \mathcal{C}$ , and let  $a, b \in \Sigma$  be such that  $a \in \Sigma(\gamma)$  and  $a \parallel b$ . We discuss different cases, observing that in all cases,  $b \neq a$  since  $a \parallel b$ .

1. Assume that  $a \leq \gamma$ . Hence  $\varphi(\gamma, a) = \gamma \setminus \{a\}$ .
  - a) If  $b \in \Sigma(\gamma)$ .
    - i. If  $b \leq \gamma$ , then  $\varphi(\gamma, b) = \gamma \setminus \{b\}$ . Since  $a \neq b$ , we also have  $a \leq \gamma \setminus \{b\}$  and thus  $a \in \Sigma(\varphi(\gamma, b))$  and  $\varphi(\varphi(\gamma, b), a) = \gamma \setminus \{a, b\}$ . For the same reason,  $b \leq \gamma \setminus \{a\}$  and thus  $b \in \Sigma(\varphi(\gamma, a))$  and  $\varphi(\varphi(\gamma, a), b) = \gamma \setminus \{a, b\} = \varphi(\varphi(\gamma, b), a)$ .
    - ii. If  $b \parallel \gamma$ , then  $\varphi(\gamma, b) = \gamma \cdot b$ . Hence  $a \leq \varphi(\gamma, b)$  and thus  $a \in \Sigma(\varphi(\gamma, b))$  and  $\varphi(\varphi(\gamma, b), a) = \gamma \cdot b \setminus \{a\}$ . We also have  $b \parallel \gamma \setminus \{a\}$  thus  $b \in \Sigma(\varphi(\gamma, a))$  and  $\varphi(\varphi(\gamma, a), b) = (\gamma \setminus \{a\}) \cdot b = \varphi(\varphi(\gamma, b), a)$ .
  - b) If  $b \notin \Sigma(\gamma)$ . Then  $b$  is not parallel to  $\gamma \setminus \{a\}$  otherwise, since  $b \parallel a$ ,  $b$  would be parallel to  $\gamma$ , contradicting  $b \notin \Sigma(\gamma)$ . And  $b \leq \gamma \setminus \{a\}$  does not hold either, otherwise  $b \leq \gamma$  would hold, contradicting  $b \notin \Sigma(\gamma)$ . We conclude that  $b \notin \Sigma(\varphi(\gamma, a))$ , as expected.
2. Assume that  $a \parallel \gamma$ . Hence  $\varphi(\gamma, a) = \gamma \cdot a$ .
  - a) If  $b \in \Sigma(\gamma)$ .
    - i. If  $b \leq \gamma$ , then  $\varphi(\gamma, b) = \gamma \setminus \{b\}$ . Therefore  $a \parallel \varphi(\gamma, b)$  thus  $a \in \Sigma(\varphi(\gamma, b))$  and  $\varphi(\varphi(\gamma, b), a) = (\gamma \setminus \{b\}) \cdot a$ . On the other hand,  $b \leq \varphi(\gamma, a)$ , thus  $b \in \Sigma(\varphi(\gamma, a))$  and  $\varphi(\varphi(\gamma, a), b) = (\gamma \cdot a) \setminus \{b\} = \varphi(\varphi(\gamma, b), a)$ .
    - ii. If  $b \parallel \gamma$ , then  $\varphi(\gamma, b) = \gamma \cdot b$ , but  $a \parallel b$  hence  $a \parallel \varphi(\gamma, b)$ , thus  $a \in \Sigma(\varphi(\gamma, b))$  and  $\varphi(\varphi(\gamma, b), a) = \gamma \cdot b \cdot a$ . On the other hand, we also have  $b \parallel \varphi(\gamma, a)$  since  $a \parallel b$ , thus  $b \in \Sigma(\varphi(\gamma, a))$  and  $\varphi(\varphi(\gamma, a), b) = \gamma \cdot a \cdot b = \varphi(\varphi(\gamma, b), a)$ .
  - b) If  $b \notin \Sigma(\gamma)$ . Then  $b \leq \varphi(\gamma, a)$  would imply  $b \leq \gamma$  since  $a \neq b$ , contradicting  $b \notin \Sigma(\gamma)$ . And  $b \parallel \varphi(\gamma, a)$  would imply  $b \parallel \gamma$ , contradicting  $b \notin \Sigma(\gamma)$ . Hence  $b \notin \Sigma(\varphi(\gamma, a))$ , as expected.

3.3.4 *Domino tilings* — Tilings of plane surfaces by  $2 \times 1$  and  $1 \times 2$  dominoes are an “extensively studied and well-understood lattice model in statistical physics and combinatorics” [23, § 1.1]. Here, we point out that domino tilings and their flips provide a vast source of examples of partial actions of trace monoids; we will focus on elementary examples only, using them for illustrative purposes.

Given a plane surface for which there exists at least one domino tiling, the flips rotate two contiguous dominoes and leave unchanged the rest of the tiling [25], as illustrated in Figure 3.

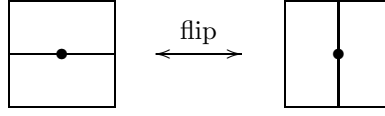


Figure 3: Illustrating the action of a flip. The dot represents the centre of the rotation.

**3.3.5 The Rabati tilings** — We define the  $n$ -Rabati strip, with  $n \geq 2$  an integer, as a strip of size  $n \times 2$ , as depicted on Figure 4 for  $n = 7$ . The in-line version is the one depicted on the figure; the circled version corresponds to the same strip, where the right most side is identified with the left most side.

Consider the in-line version of the Rabati strip first. Then associate the trace monoid with  $n - 1$  generators, say  $\Sigma = \{a_1, \dots, a_{n-1}\}$ , and independence relation  $I$  defined by:

$$I = \{(a_i, a_j) : |i - j| \geq 2\}.$$

For the circle version of the Rabati strip, we consider the trace monoid with  $n$  generators,  $\Sigma = \{a_0, \dots, a_{n-1}\}$ , and independence relation:

$$I = \{(a_i, a_j) : (|i - j| \bmod n) \geq 2\}$$

In both cases, the following facts are elementary (see Figure 4):

1. The domino tilings of the Rabati strip are in bijection with the cliques of the associated trace monoid. The empty clique corresponds to the tiling where all dominoes are in vertical position.
2. Let  $\gamma \in \mathcal{C}$  correspond to some tiling of the strip. The flip centred on the  $i^{\text{th}}$  position is enabled if and only if the tip-top action  $\varphi(\gamma, a_i)$  is enabled; and the result  $\varphi(\gamma, a_i)$  of the tip-top action corresponds to the tiling resulting from the action of this flip on the tiling  $\gamma$ .

Hence, for tilings of Rabati strips, the action of flips corresponds exactly to the tip-top action of the associated trace monoid.

For tilings of general plane surfaces, we can still define a trace monoid associated to the action of flips, where each flip corresponds to a generator of the monoid, and commutativity of generators corresponds to flips that operate on disjoint pairs of dominoes. The combinatorial action of flips is encoded in the action of this trace monoid on the set of tilings. But this action does not correspond in general to the tip-top action of the trace monoid—it is, in general, much more complex.

### 3.4 — Sub-actions and irreducibility

We borrow some usual terminology and notions from sub-shift theory [27, 20] to qualify states and define irreducibility classes. Throughout this subsection, we fix a partial action  $X \times \mathcal{M} \rightarrow X$  of the trace monoid  $\mathcal{M} = \mathcal{M}(\Sigma, I)$  over the set  $X$ .



Figure 4: The in-line  $n$ -Rabati strip with  $n = 7$ . The tiling on the left corresponds to the empty clique. The tiling on the right correspond to the clique  $a_1 \cdot a_6$ .

**3.4.1 Communicating states and essential states** — A state  $\alpha \neq \perp$  leads to a state  $\beta$ , which we denote by  $\alpha \Rightarrow \beta$ , if there exists a non empty trace  $x \in \mathcal{M}_\alpha \setminus \{1\}$  such that  $\alpha \cdot x = \beta$  (and thus  $\beta \neq \perp$ ). Two states  $\alpha$  and  $\beta$  communicate if both  $\alpha \Rightarrow \beta$  and  $\beta \Rightarrow \alpha$  hold, which we denote by  $\alpha \Leftrightarrow \beta$ . A state  $\alpha$  is essential if  $\alpha \neq \perp$  and holds:  $\forall \beta \in X \quad \alpha \Rightarrow \beta$  implies  $\alpha \Leftrightarrow \beta$ .

Note that, by the action property, the relation  $\alpha \Rightarrow \beta$  is transitive on  $X \setminus \{\perp\}$ . Also, since  $\Sigma(\alpha) \neq \emptyset$  according to Definition 3.3, every state leads to at least one state.

• **Proposition 3.7**—*There exists at least one essential state.*

*Proof.* Seeking a contradiction, assume there is no essential state. Pick  $\alpha_1 \in X \setminus \{\perp\}$ , which is a non empty set by definition. Then, by induction, construct an infinite sequence of states  $(\alpha_i)_{i \geq 1}$  such that:

$$\forall i \geq 1 \quad \alpha_i \Rightarrow \alpha_{i+1} \wedge \neg(\alpha_{i+1} \Rightarrow \alpha_i).$$

Then, by transitivity of the communicating relation, holds:

$$\forall i < j \quad \alpha_i \Rightarrow \alpha_j \wedge \neg(\alpha_j \Rightarrow \alpha_i).$$

Since  $X$  is a finite set, there are two integers  $i < j$  such that  $\alpha_i = \alpha_j$ . But then  $\alpha_i \Rightarrow \alpha_i$  and  $\neg(\alpha_i \Rightarrow \alpha_i)$  both hold, a contradiction.  $\square$

**3.4.2 Sub-actions and irreducible sub-actions** — A non empty subset  $Y \subseteq X \setminus \{\perp\}$  defines a sub-action if  $\alpha \cdot a \in Y$  for all  $\alpha \in Y$  and for all  $a \in \Sigma(\alpha)$ . Equivalently,  $\alpha \cdot x \in Y$  for all  $\alpha \in Y$  and for all  $x \in \mathcal{M}_\alpha$ . In this case, the restriction of  $X \times \mathcal{M} \rightarrow X$  to  $(Y \cup \{\perp\}) \times \mathcal{M} \rightarrow Y \cup \{\perp\}$  defines a partial action. Clearly, sub-actions correspond to non empty subsets of  $X \setminus \{\perp\}$  closed under the communicating relation between states.

We define a sub-action  $Y$  to be irreducible if it has no other sub-action than itself, and an action to be irreducible if it is irreducible as a sub-action of itself. Clearly, every state of an irreducible sub-action is essential in the sub-action and in the original action.

• **Proposition 3.8**—*The set of essential states of the action defines a sub-action. For each essential state  $\alpha$ , the set*

$$X_\alpha = \{\beta \in X : \alpha \Rightarrow \beta\}$$

*defines an irreducible sub-action. Every irreducible sub-action is of this form.*

*Proof.* The transitivity property of the communicating relation implies that any essential state leads to an essential state. Hence the set of essential states, which is non empty by Proposition 3.7, is closed under the communicating relation between states and defines thus a sub-action.

Let  $\alpha \in X$  be an essential state. Then  $X_\alpha$  is the strongly connected component containing  $\alpha$  of the communicating relation. It is thus irreducible. Conversely, if  $Y$  is an irreducible sub-action, pick  $\alpha \in Y$ . Then  $X_\alpha \subseteq Y$  by definition of  $Y$  being a sub-action, and thus  $X_\alpha = Y$  by definition of  $Y$  being irreducible.  $\square$

**3.4.3 Example** — The tip-top action  $\varphi : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{C}$  of a trace monoid over its set of cliques (§ 3.3.3) is irreducible, and thus every clique is an essential state. Indeed, if  $\gamma \neq 1$ , then  $\varphi(\gamma, \gamma) = 1$ ; and for  $\gamma = 1$ , pick any letter  $a \in \Sigma$ , then  $\varphi(\varphi(1, a), a) = \varphi(a, a) = 1$ . Henceforth every clique leads to the empty clique. And similarly it is easy to see that the empty clique leads to every clique. We deduce  $\gamma \Leftrightarrow \gamma'$  for all cliques  $\gamma, \gamma'$ , and thus the action is irreducible.

## 4—Markov measures

We now come to our main object of study, Markov measures associated with actions of trace monoids. In this section, we examine Markov measures associated with total actions. Next section is devoted to additional topics of study when the total action actually originates from a partial action.

### 4.1 — Markov measures and fibred valuations

4.1.1 *Definitions* — We recall that the boundary  $\partial\mathcal{M}$  of a trace monoid  $\mathcal{M}$  (§ 2.3.2) is equipped with its Borel  $\sigma$ -algebra (§ 2.4.1).

• **Definition 4.1**—Let  $X \times \mathcal{M} \rightarrow X$  be a total action of a trace monoid  $\mathcal{M}$  over a finite set  $X$ . A Markov measure associated with this action is a family  $\mathbb{P} = (\mathbb{P}_\alpha)_{\alpha \in X}$ , where each  $\mathbb{P}_\alpha$  is a probability measure on the boundary  $\partial\mathcal{M}$ , satisfying the following chain rule:

$$\forall \alpha \in X \quad \forall x, y \in \mathcal{M} \quad \mathbb{P}_\alpha(\uparrow(x \cdot y)) = \mathbb{P}_\alpha(\uparrow x) \mathbb{P}_{\alpha \cdot x}(\uparrow y). \quad (6)$$

A fibred valuation associated with this action is a family  $F = (f_\alpha)_{\alpha \in X}$ , where each  $f_\alpha : \mathcal{M} \rightarrow \mathbb{R}$  is a real-valued mapping, with the following property:

$$\forall \alpha \in X \quad f_\alpha(1) = 1 \quad (7)$$

$$\forall \alpha \in X \quad \forall x, y \in \mathcal{M} \quad f_\alpha(x \cdot y) = f_\alpha(x) f_{\alpha \cdot x}(y) \quad (8)$$

If  $\mathbb{P} = (\mathbb{P}_\alpha)_{\alpha \in X}$  is a Markov measure, the family  $F = (f_\alpha)_{\alpha \in X}$  defined by:

$$\forall \alpha \in X \quad \forall x \in \mathcal{M} \quad f_\alpha(x) = \mathbb{P}_\alpha(\uparrow x) \quad (9)$$

is called the fibred valuation associated with  $\mathbb{P}$ . The support valuation of  $\mathbb{P}$  is the family  $S = (s_\alpha)_{\alpha \in X}$  of mappings  $s_\alpha : \mathcal{M} \rightarrow \{0, 1\}$  defined by:

$$\forall \alpha \in X \quad \forall x \in \mathcal{M} \quad s_\alpha(x) = \begin{cases} 1, & \text{if } \mathbb{P}_\alpha(\uparrow x) > 0, \\ 0, & \text{if } \mathbb{P}_\alpha(\uparrow x) = 0. \end{cases} \quad (10)$$

4.1.2 *Elementary remarks* — Bernoulli measures on  $\partial\mathcal{M}$  (§ 2.4.3) correspond to Markov measures  $\mathbb{P} = (\mathbb{P}_\alpha)_{\alpha \in X}$  such that  $\mathbb{P}_\alpha$  is independent of  $\alpha$ . In particular, Markov measures exist.

If  $\mathcal{M} = \Sigma^*$  is the free monoid with its natural right action on  $\Sigma$ , then Markov measures correspond to Markov chains on  $\Sigma$ . Bernoulli measures correspond to Bernoulli sequences with values in  $\Sigma$ .

Clearly, the support valuation of a Markov measure is a fibred valuation.

Since the family of elementary cylinders forms a  $\pi$ -system (§ 2.4.1) of the Borel  $\sigma$ -algebra on  $\partial\mathcal{M}$ , a Markov measure  $\mathbb{P} = (\mathbb{P}_\alpha)_{\alpha \in X}$  is entirely characterised by the countable family of non negative numbers  $\{\mathbb{P}_\alpha(\uparrow x) : \alpha \in X, x \in \mathcal{M}\}$ . In turn, by the chain rule (6), these are entirely determined by the finite family of non negative numbers  $\{f_\alpha(a) : \alpha \in X, a \in \Sigma\}$ . Hence, characterising Markov measures consists in finding adequate normalisation conditions on finite families of non negative numbers of the form  $(\lambda_\alpha(a))_{(\alpha, a) \in X \times \Sigma}$ .

4.1.3 *Fibred valuations* — A first task consists in elucidating the—rather simple—structure of fibred valuations.

• **Proposition 4.2**—Fibred valuations are in bijection with families  $(\lambda_\alpha(a))_{(\alpha, a) \in X \times \Sigma}$  of real numbers satisfying the following series of equations, that we call the concurrency equations:

$$\forall \alpha \in X \quad \forall a, b \in \Sigma \quad a \parallel b \implies \lambda_\alpha(a) \lambda_{\alpha \cdot a}(b) = \lambda_\alpha(b) \lambda_{\alpha \cdot b}(a) \quad (11)$$

The bijection associates to a fibred valuation  $(f_\alpha)_{\alpha \in X}$  the family of reals  $(f_\alpha(a))_{(\alpha, a) \in X \times \Sigma}$ .

*Proof.* If  $(f_\alpha)_{\alpha \in X}$  is a fibred valuation, then the family of reals defined by  $\lambda_\alpha(a) = f_\alpha(a)$  satisfies the concurrency equations, since both the left and the right members of the equation in (11) represent  $f_\alpha(a \cdot b)$ , when  $a \parallel b$ .

Conversely, let  $(\lambda_\alpha(a))_{(\alpha,a) \in X \times \Sigma}$  be a family of reals satisfying the concurrency equations. We show the existence of a fibred valuation  $F = (f_\alpha)_{\alpha \in X}$  satisfying  $f_\alpha(a) = \lambda_\alpha(a)$  for all  $\alpha \in X$  and  $a \in \Sigma$ .

Define for each  $\alpha \in X$  and each word  $x = a_1 \dots a_n$  of  $\Sigma^*$ :

$$f_\alpha(x) = \lambda_\alpha(a_1) \lambda_{\alpha \cdot a_1}(a_2) \cdots \lambda_{\alpha \cdot a_1 \dots a_{n-1}}(a_n), \quad (12)$$

with  $f_\alpha(1) = 1$  by convention. Then the concurrency equations imply that  $f_\alpha(x) = f_\alpha(y)$  for any word  $y$  which is in immediate equivalence with  $x$  (§ 2.1.4). Therefore  $f_\alpha$  factorises through a mapping, still denoted  $f_\alpha : \mathcal{M} \rightarrow \mathbb{R}$  such that  $f_\alpha(a) = \lambda_\alpha(a)$  for all  $a \in \Sigma$ . The definition (12) implies the validity of the chain rule (8), whence the sought fibred valuation  $F = (f_\alpha)_{\alpha \in X}$ .  $\square$

• **Corollary 4.3**—*If  $(\mathbb{P}_\alpha)_{\alpha \in X}$  is a Markov measure, then the numbers  $\lambda_\alpha(a) = \mathbb{P}_\alpha(\uparrow a)$ , for  $\alpha$  ranging over  $X$  and  $a$  ranging over  $\Sigma$ , satisfy the concurrency equations (11).*

## 4.2 — Characterisation of Markov measures

Our aim is to give necessary and sufficient conditions for a family  $(\lambda_\alpha(a))_{(\alpha,a) \in X \times \Sigma}$  of real numbers to correspond to the family  $(f_\alpha(a))_{(\alpha,a) \in X \times \Sigma}$  obtained from the fibred valuation of a Markov measure. Theorem 4.5 below gives necessary conditions, postponing their sufficiency to next subsection.

4.2.1 *Möbius fibred valuations* — The following definition generalises to fibred valuations the notion of Möbius valuation recalled in § 2.4.4.

• **Definition 4.4**—*Let  $F = (f_\alpha)_{\alpha \in X}$  be a fibred valuation. For each  $\alpha \in X$ , let  $h_\alpha : \mathcal{C} \rightarrow \mathbb{R}$  be the Möbius transform (§ 2.2.4) of  $f_\alpha : \mathcal{C} \rightarrow \mathbb{R}$ . The fibred valuation  $F$  is Möbius if it satisfies the following conditions, for all  $\alpha \in X$ :*

$$h_\alpha(1) = 0, \quad (13)$$

$$\forall \gamma \in \mathcal{C} \quad h_\alpha(\gamma) \geq 0. \quad (14)$$

Remark that, as a consequence of the second Möbius inversion formula (§ 2.2.6), a Möbius valuation is necessarily non negative.

4.2.2 *Identification theorem* — The purpose of the following theorem is twofold. First, it provides necessary conditions on families of numbers that generate a Markov measure. Second, it shows that the random process of “states-and-cliques” (defined in the statement) has the structure of a Markov chain, of which the initial measure and the transition matrix are entirely determined by the Markov measure. In both aspects, it generalises the corresponding results for Bernoulli measures (§ 2.4.5 and § 2.4.7).

• **Theorem 4.5**—*Let  $\mathbb{P} = (\mathbb{P}_\alpha)_{\alpha \in X}$  be a Markov measure on a total action  $X \times \mathcal{M} \rightarrow X$ . Then:*

1. *The fibred valuation associated with  $\mathbb{P}$  is Möbius.*
2. *Let  $(C_i)_{i \geq 1}$  and  $(Y_i)_{i \geq 0}$  be the two random sequences introduced in § 2.4.6, and for each integer  $i \geq 0$ , let  $X_i = \alpha \cdot Y_i$ . Then, under the probability  $\mathbb{P}_{\alpha_0}$ , the sequence  $(X_{i-1}, C_i)_{i \geq 1}$  is a homogeneous Markov chain with values in  $X \times \mathcal{C}$ .*
3. *The initial measure of the Markov chain  $(X_{i-1}, C_i)_{i \geq 1}$  is  $\delta_{\{\alpha_0\}} \otimes (h_{\alpha_0}|_{\mathcal{C}})$ , where  $h_{\alpha_0}|_{\mathcal{C}} : \mathcal{C} \rightarrow \mathbb{R}$  denotes the restriction of  $h_{\alpha_0}$  to non empty cliques, and  $h_{\alpha_0} : \mathcal{C} \rightarrow \mathbb{R}$  is the Möbius transform of the fibred valuation associated with  $\mathbb{P}$ .*



4. The transition matrix of the chain is independent of  $\alpha_0$ . It is given by:

$$P_{(\alpha, \gamma), (\alpha', \gamma')} = \mathbf{1}_{\{\alpha' = \alpha \cdot \gamma\}} \mathbf{1}_{\{\gamma \rightarrow \gamma'\}} \frac{h_{\alpha'}(\gamma')}{g_{\alpha'}(\gamma)}, \quad (15)$$

where  $\gamma \rightarrow \gamma'$  denotes the Cartier-Foata relation between cliques (§ 2.1.10), and  $g_{\alpha'} : \mathfrak{C} \rightarrow \mathbb{R}$  is the non negative function defined by:

$$\forall \gamma \in \mathfrak{C} \quad g_{\alpha'}(\gamma) = \sum_{\gamma' \in \mathfrak{C} : \gamma \rightarrow \gamma'} h_{\alpha'}(\gamma'). \quad (16)$$

If  $g_{\alpha'}(\gamma) = 0$ , then the expression (15) is replaced by:

$$P_{(\alpha, \gamma), (\alpha', \gamma')} = 0.$$

**4.2.3 Remark on the transition matrix** — The matrix  $P$  described above might have some lines entirely filled with zeros. This happens for the lines indexed by pairs  $(\alpha, \gamma)$  where  $\gamma$  is such that  $g_{\alpha \cdot \gamma}(\gamma) = 0$ .

Although formally forbidden for stochastic matrices, this is actually of little inconvenience since these lines correspond to states that will never be reached.

**4.2.4 Strategy of proof** — We first establish the two following key lemmas. All the statements of Theorem 4.5 derive from them.

The proof follows closely the same line as the proof of the corresponding results for Bernoulli measures (§ 2.4.5 and § 2.4.7), with the specific issue here consisting in correctly dealing with the random current state.

• **Lemma 4.6**—Let  $(\mathbb{P}_\alpha)_{\alpha \in X}$  be a Markov measure associated with a total action  $X \times \mathcal{M} \rightarrow X$ , and with fibred valuation  $(f_\alpha)_{\alpha \in X}$ . For each  $\alpha \in X$ , let  $h_\alpha : \mathcal{M} \rightarrow \mathbb{R}$  be the graded Möbius transform (§ 2.2.5) of  $f_\alpha(\cdot)$ . Let  $n \geq 1$  be an integer, and let  $x \in \mathcal{M}$  be a trace of height  $\tau(x) = n$ . Then:

$$\mathbb{P}_\alpha(Y_n = x) = h_\alpha(x). \quad (17)$$

*Proof.* Let  $x$  and  $n$  be as in the statement. The formula (5) decomposes the elementary cylinder  $\uparrow x$  as a disjoint union, whence:

$$\mathbb{P}_\alpha(\uparrow x) = \sum_{z \in \mathcal{M}(x)} \mathbb{P}_\alpha(Y_n = z).$$

We also have  $\mathbb{P}_\alpha(\uparrow x) = f_\alpha(x)$  by definition of  $f_\alpha$ . Hence, by the reciprocal of the second Möbius inversion formula (§ 2.2.6), we deduce that the two functions  $f_\alpha$  and  $x \in \mathcal{M} \mapsto \mathbb{P}_\alpha(Y_{\tau(x)} = x)$  are related by the graded Möbius formula, whence (17).  $\square$

• **Lemma 4.7**—Let  $F = (f_\alpha)_{\alpha \in X}$  be a fibred valuation. For each  $\alpha \in X$ , let  $h_\alpha : \mathfrak{C} \rightarrow \mathbb{R}$  be the Möbius transform of  $f_\alpha$ , and let  $g_\alpha : \mathfrak{C} \rightarrow \mathbb{R}$  be defined as in (16). Assume that  $h_\alpha(1) = 0$  for all  $\alpha \in X$  (in particular, this holds if  $F$  is Möbius).

Then, for all  $\beta \in X$  and for all  $\gamma \in \mathfrak{C}$ , holds:

$$f_\beta(\gamma) g_\alpha(\gamma) = h_\beta(\gamma), \quad \text{for } \alpha = \beta \cdot \gamma. \quad (18)$$

*Proof.* Let  $\beta \in X$  and  $\gamma \in \mathfrak{C}$ , and put  $\alpha = \beta \cdot \gamma$ . We compute as follows:

$$\begin{aligned} g_\alpha(\gamma) &= \sum_{\gamma' \in \mathfrak{C} : \gamma \rightarrow \gamma'} h_\alpha(\gamma') \\ &= \sum_{\gamma' \in \mathfrak{C} : \gamma \rightarrow \gamma'} \left( \sum_{\gamma'' \in \mathfrak{C} : \gamma' \geq \gamma''} (-1)^{|\gamma''| - |\gamma'|} f_\alpha(\gamma'') \right) \\ &= \sum_{\gamma'' \in \mathfrak{C}} (-1)^{|\gamma''|} f_\alpha(\gamma'') B(\gamma, \gamma'') \end{aligned}$$

with

$$B(\gamma, \gamma'') = \sum_{\gamma' \in \mathfrak{C}} (-1)^{|\gamma'|} \mathbf{1}_{\{\gamma \rightarrow \gamma'\}} \mathbf{1}_{\{\gamma' \leq \gamma''\}}.$$

For  $\gamma, \gamma'' \in \mathcal{C}$ , a clique  $\gamma' \in \mathfrak{C}$  satisfies  $\gamma \rightarrow \gamma'$  and  $\gamma' \leq \gamma''$  if and only if  $\gamma'' \leq \delta$ , where  $\delta = \{a \in \gamma'' : \gamma \rightarrow a\}$ . Therefore, adding and subtracting the empty clique in the sum defining  $B(\gamma, \gamma'')$  and using the binomial formula, we find:

$$B(\gamma, \gamma'') = \sum_{\gamma' \in \mathfrak{C} : \gamma' \leq \delta} (-1)^{|\gamma'|} - 1 = \mathbf{1}_{\{\delta=1\}} - 1 = -\mathbf{1}_{\{\neg(\gamma'' \parallel \gamma)\}}.$$

We obtain thus the following expression for  $g_\alpha(\gamma)$ :

$$g_\alpha(\gamma) = - \sum_{\gamma'' \in \mathcal{C} : \neg(\gamma'' \parallel \gamma)} (-1)^{|\gamma''|} f_\alpha(\gamma''). \quad (19)$$

Writing down the hypothesis  $h_\alpha(1) = 0$  yields, by the very definition of the Möbius transform  $h_\alpha$ :

$$\sum_{\gamma'' \in \mathcal{C}} (-1)^{|\gamma''|} f_\alpha(\gamma'') = 0.$$

Separating this sum according to those cliques  $\gamma''$  which are on the one hand, and which are not on the other hand, parallel to  $\gamma$ , and by virtue of (19), we obtain:

$$g_\alpha(\gamma) = \sum_{\gamma'' \in \mathcal{C} : \gamma'' \parallel \gamma} (-1)^{|\gamma''|} f_\alpha(\gamma''). \quad (20)$$

Since  $\alpha = \beta \cdot \gamma$  on the one hand, and by the chain rule for fibred valuations on the other hand, we have:

$$f_\beta(\gamma \cdot \gamma'') = f_\beta(\gamma) f_\alpha(\gamma'').$$

Therefore, multiplying both sides of (20) by  $f_\beta(\gamma)$  yields:

$$\begin{aligned} f_\beta(\gamma) g_\alpha(\gamma) &= \sum_{\gamma'' \parallel \gamma} (-1)^{|\gamma''|} f_\beta(\gamma \cdot \gamma'') \\ &= \sum_{\delta \in \mathcal{C} : \delta \geq \gamma} (-1)^{|\delta| - |\gamma|} f_\beta(\delta) \\ &= h_\beta(\gamma), \end{aligned}$$

which was to be proved.  $\square$

**4.2.5 Proof of Theorem 4.5** — Let us show that the fibred valuation  $F = (f_\alpha)_{\alpha \in X}$  associated with the Markov measure  $\mathbb{P}$  is Möbius. According to Lemma 4.6, and since  $C_1 = Y_1$ , we have:

$$\forall \gamma \in \mathfrak{C} \quad \mathbb{P}_\alpha(C_1 = \gamma) = h_\alpha(\gamma).$$

It follows at once that  $h_\alpha(\gamma) \geq 0$  for all  $\gamma \in \mathfrak{C}$ , which proves the property (14). And applying the total probability law to the first clique  $C_1$  yields:

$$\sum_{\gamma \in \mathfrak{C}} h_\alpha(\gamma) = 1.$$

But we also have, by the Möbius inversion formula (3):

$$\begin{aligned} f_\alpha(1) &= h_\alpha(1) + \sum_{\gamma \in \mathfrak{C}} h_\alpha(\gamma) \\ 1 &= h_\alpha(1) + 1, \end{aligned}$$

whence  $h_\alpha(1) = 0$ , which proves the property (13). Henceforth we have proved that  $F$  is Möbius and that the law of  $(X_0, C_1)$  under  $\mathbb{P}_\alpha$  is indeed given by  $\delta_{\{\alpha\}} \otimes (h_\alpha|_{\mathfrak{C}})$ . This proves points 1 and 3 of the theorem.

Let  $P$  be the square matrix indexed by  $(X \times \mathfrak{C})^2$  and defined by (15), with the restriction that  $P_{(\alpha, \gamma), (\alpha', \gamma')} = 0$  whenever  $g_\alpha(\gamma) = 0$ . We shall prove that, for any sequence  $(x_0, c_1), \dots, (x_{n-1}, c_n)$  in  $X \times \mathfrak{C}$  with  $n \geq 1$ , if  $p_n$  denotes the probability

$$p_n = \mathbb{P}_\alpha((X_0, C_1) = (x_0, c_1), \dots, (X_{n-1}, C_n) = (x_{n-1}, c_n))$$

and if  $q_n$  denotes the result of the chain rule:

$$q_n = \mathbf{1}_{\{\alpha=x_0\}} h_{x_0}(c_1) P_{(x_0, c_1), (x_1, c_2)} \cdots P_{(x_{n-2}, c_{n-1}), (x_{n-1}, c_n)}$$

then  $p_n = q_n$ . This will prove the statements 2 and 4 and complete the proof Theorem 4.5.

First, it is clear that  $p_n = q_n = 0$  if any of the following statements does not hold:

$$x_0 = \alpha, \tag{21}$$

$$\forall i \in \{0, \dots, n-2\} \quad x_i \cdot c_{i+1} = x_{i+1}, \tag{22}$$

$$\forall i \in \{1, \dots, n-1\} \quad c_i \rightarrow c_{i+1}. \tag{23}$$

Hence, assuming that all the above statements (21)–(23) hold, we have:

$$\begin{aligned} p_n &= \mathbb{P}_\alpha(Y_n = c_1 \cdots c_n) \\ &= h_\alpha(c_1 \cdots c_n) \end{aligned} \quad \text{by Lemma 4.6}$$

Defining  $y_j = c_1 \cdots c_j$  for all  $j \in \{0, \dots, n\}$ , the definition of the graded Möbius transform (§ 2.2.5) together with the chain rule relation for fibred valuations yield:

$$p_n = f_\alpha(y_{n-1}) h_{x_{n-1}}(c_n) \tag{24}$$

The above implies in particular the following inequality, valid for any integer  $j \leq n$ :

$$\begin{aligned} p_n &\leq \mathbb{P}_\alpha(Y_j = c_1 \cdots c_j) \\ &\leq f_\alpha(y_{j-1}) h_{x_{j-1}}(c_j) \end{aligned}$$

Therefore, if  $g_{x_{j-1}}(c_{j-1}) = 0$  for some integer  $j \in \{2, \dots, n\}$ , then on the one hand,  $h_{x_{j-1}}(\gamma) = 0$  for all  $\gamma \in \mathfrak{C}$  such that  $c_{j-1} \rightarrow \gamma$  holds, and in particular  $h_{x_{j-1}}(c_j) = 0$ , and thus  $p_n = 0$ . But  $q_n = 0$  also on the other hand, and thus  $p_n = q_n$ .

It remains thus to examine the most interesting case, where all statements (21)–(23) hold and  $g_{x_{j-1}}(c_j) \neq 0$  for all  $j \in \{2, \dots, n\}$ . In this case, we apply the result of Lemma 4.7 and we divide by  $g_{x_{j-1}}(c_j)$  to obtain, since  $x_j = x_{j-1} \cdot c_j$ :

$$\forall j \in \{1, \dots, n-1\} \quad f_{x_{j-1}}(c_j) = \frac{h_{x_{j-1}}(c_j)}{g_{x_j}(c_j)} \tag{25}$$

We evaluate  $q_n$ :

$$\begin{aligned}
q_n &= h_{x_0}(c_1) \frac{h_{x_1}(c_2)}{g_{x_1}(c_1)} \cdots \frac{h_{x_{n-1}}(c_n)}{g_{x_{n-1}}(c_{n-1})} \\
&= \frac{h_{x_0}(c_1)}{g_{x_1}(c_1)} \frac{h_{x_1}(c_2)}{g_{x_2}(c_2)} \cdots \frac{h_{x_{n-2}}(c_{n-1})}{g_{x_{n-1}}(c_{n-1})} h_{x_{n-1}}(c_n) \\
&= f_{x_0}(c_1) f_{x_1}(c_2) \cdots f_{x_{n-2}}(c_{n-1}) h_{x_{n-1}}(c_n) && \text{by (25)} \\
&= f_\alpha(c_1 \cdots c_{n-1}) h_{x_{n-1}}(c_n) && \text{by the chain rule} \\
&= p_n && \text{by (24)}
\end{aligned}$$

This completes the proof of Theorem 4.5.  $\square$

### 4.3 — Realisation of Markov measures

4.3.1 *A reciprocal to Theorem 4.5* — A striking result of Theorem 4.5 is that the fibred valuation associated with a Markov measure is necessarily Möbius. The next result shows that the converse is true. Furthermore, it provides a realisation of the associated Markov measure by means of the Markov chain of states-and-cliques.

The effectiveness of the Möbius conditions for fibred valuations, and thus for the parameters of Markov measures, is discussed below in § 4.3.2.

• **Theorem 4.8**—*Let  $F = (f_\alpha)_{\alpha \in X}$  be a Möbius fibred valuation. Then there exists a unique Markov measure  $\mathbb{P} = (\mathbb{P}_\alpha)_{\alpha \in X}$  such that  $f_\alpha(x) = \mathbb{P}_\alpha(\uparrow x)$  for all  $\alpha \in X$  and all  $x \in \mathcal{M}$ .*

*Let  $P$  be the transition matrix on  $X \times \mathfrak{C}$  defined in (15). For each  $\alpha \in X$ , let  $(X_{i-1}, C_i)_{i \geq 1}$  be a Markov chain with transition matrix  $P$  and with initial measure  $\delta_{\{\alpha\}} \otimes (h_\alpha|_{\mathfrak{C}})$ , defined on the canonical probability space  $(\Omega, \mathfrak{G}, \mathbb{Q}_\alpha)$ . Then  $\mathbb{P}_\alpha$  is the law of the random variable  $\xi : \Omega \rightarrow \partial\mathcal{M}$  defined by:*

$$\xi = \bigvee_{n \geq 1} (C_1 \cdots C_n),$$

*under the probability  $\mathbb{Q}_\alpha$ .*

*Proof.* We first need to check that the Markov chain in the statement is well defined, that is to say, that  $\delta_{\{\alpha\}} \otimes (h_\alpha|_{\mathfrak{C}})$  is a probability distribution on  $X \times \mathfrak{C}$ , and that  $P$  is a stochastic matrix. The second point is obvious by construction. For the first point, indeed  $h_\alpha|_{\mathfrak{C}}$  is a probability distribution on  $\mathfrak{C}$  since, on the one hand,  $h_\alpha \geq 0$  by definition of  $F$  being Möbius, and on the other hand, by the second Möbius inversion formula (§ 2.2.6):

$$\begin{aligned}
\sum_{\gamma \in \mathfrak{C}} h_\alpha(\gamma) &= \sum_{\gamma \in \mathfrak{C}} h_\alpha(\gamma) && \text{since } h_\alpha(1) = 0 \\
&= f_\alpha(1) = 1.
\end{aligned}$$

For each  $\alpha \in X$ , let  $\mathbb{P}_\alpha$  be the probability measure on  $\partial\mathcal{M}$  defined as in the statement as the law of  $\bigvee_{n \geq 1} (C_1 \cdots C_n)$ , under  $\mathbb{Q}_\alpha$ . Let  $x \in \mathcal{M}$  be a trace. The decomposition from § 2.4.9 of the elementary cylinder  $\uparrow x$  yields:

$$\mathbb{P}_\alpha(\uparrow x) = \sum_{z \in \mathcal{M}(x)} \mathbb{Q}_\alpha(C_1 \cdots C_{\tau(x)} = z). \quad (26)$$

Let  $n = \tau(x)$ , and  $z \in \mathcal{M}$  be a trace of height  $\tau(z) = n$ . Then  $z$  has a Cartier-Foata decomposition of the form  $c_1 \rightarrow \dots \rightarrow c_n$ . We set  $y_j = c_1 \dots c_j$  and  $x_j = \alpha \cdot y_j$  for all integers  $j \in \{0, \dots, n\}$ , and:

$$q_n = \mathbb{Q}_\alpha(C_1 = c_1, \dots, C_n = c_n).$$

Then we have:

$$\begin{aligned} q_n &= \mathbb{Q}_\alpha(X_0 = \alpha, C_1 = c_1, \dots, X_{n-1} = \alpha_{n-1}, C_n = c_n) \\ &= h_\alpha(c_1) \frac{h_{x_1}(c_2)}{g_{x_1}(c_1)} \dots \frac{h_{x_{n-1}}(c_n)}{g_{x_{n-1}}(c_{n-1})} \\ &= \frac{h_{x_0}(c_1)}{g_{x_1}(c_1)} \dots \frac{h_{x_{n-2}}(c_{n-1})}{g_{x_{n-1}}(c_{n-1})} h_{x_{n-1}}(c_n) \quad \text{since } \alpha = x_0 \\ &= f_{x_0}(c_1) \dots f_{x_{n-2}}(c_{n-1}) h_{x_{n-1}}(c_n), \end{aligned}$$

the last equality by using Lemma 4.7, which applies since  $F$  is Möbius by assumption.

By the chain rule for the fibred valuation  $F$  on the one hand, and according to the definition of the graded Möbius transform (§ 2.2.5) on the other hand, we obtain thus:  $q_n = h_\alpha(z)$ .

Returning to (26), applying the second Möbius inversion formula (§ 2.2.6) yields:

$$\mathbb{P}_\alpha(\uparrow x) = \sum_{z \in \mathcal{M}(x)} h_\alpha(z) = f_\alpha(x).$$

This completes the proof of the theorem.  $\square$

**4.3.2 Effectiveness of Möbius conditions** — Since Theorems 4.5 and 4.8 entirely characterise Markov measures by means of Möbius fibred valuations, it is natural to examine to which extent this condition is effective.

A natural question is the following: given a family of real numbers  $\lambda = (\lambda_\alpha(a))_{(\alpha,a) \in X \times \Sigma}$ , can we effectively determine the existence of a Markov measure  $\mathbb{P} = (\mathbb{P}_\alpha)_{\alpha \in X}$  such that  $\mathbb{P}_\alpha(\uparrow a) = \lambda_\alpha(a)$  for all  $(\alpha, a) \in X \times \Sigma$ ? It amounts to knowing whether:

1. The family  $\lambda$  defines a fibred valuation.
2. In this case, the fibred valuation has to be Möbius.

The first point is solved by Proposition 4.2: the family  $\lambda$  must satisfy the concurrency equations. These are a finite number of equalities to be satisfied.

For the second point, we observe that the Möbius conditions stated in Definition 4.4 consist in a series of equalities and another series of inequalities. Assuming that the first point is fulfilled, the equalities and the inequalities only involve polynomial expressions of the terms  $\lambda_\alpha(a)$ , for  $(\alpha, a)$  ranging over  $X \times \Sigma$ .

Finally, the question is entirely answered by means of polynomial conditions involving only the terms of the family  $\lambda$ . Giving a parametric form for all Möbius fibred valuations, and thus all Markov measures, is yet another question which is not treated here.

Next subsection is devoted to the study of an example illustrating the use of the machinery described above.

## 4.4 — Examples of Markov measures on a Rabati tiling

**4.4.1 Setting** — We consider the 4-Rabati strip in line (§ 3.3.5), of which we depict in Figure 5 the five possible domino tilings. Recall that the tilings correspond to the set of five cliques  $\mathcal{C} = \{1, a, b, c, a \cdot c\}$  of the trace monoid

$$\mathcal{M} = \langle a, b, c \mid ac = ca \rangle.$$

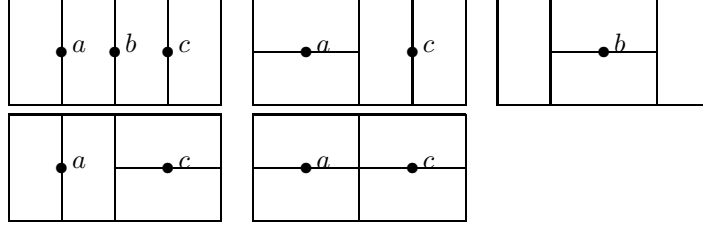


Figure 5: The five tilings of the 4-Rabati strip corresponding, from left to right, to the cliques 1,  $a$ ,  $b$  on the top line and to the cliques  $c$  and  $a \cdot c$  on the bottom line.

The partial action generated by the enabled flips, which corresponds to the tip-top action of the trace monoid  $\mathcal{M}$  (§ 3.3.3), is conveniently represented by a graph, depicted on Figure 6–(a). Since each action is reversible, the graph is non oriented.

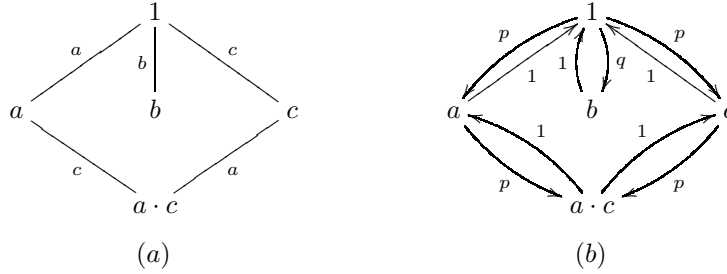


Figure 6: (a)—Graph of the action of flips on the 4-Rabati tilings. Vertices are labelled by states (cliques) and edges are labelled by elementary actions (letters). (b)—Graph of the probabilistic actions of flips on the same Rabati tiling, with probabilistic parameters  $q$  and  $p = 1 - \sqrt{q}$ .

4.4.2 *Markov measures for the partial action* — In order to fit with the previous setting, we have to consider the total action  $(\mathcal{C} \cup \{\perp\}) \times \mathcal{M} \rightarrow (\mathcal{C} \times \{\perp\})$ , as described in § 3.2.2. Introducing the probabilistic dynamics however, we impose the additional constraint that  $\mathbb{P}_\gamma(\uparrow \perp) = 0$  for all  $\gamma \in \mathcal{C}$ , so that  $\perp$  is never reached. In the following, it amounts to merely omitting  $\perp$ .

4.4.3 *Markov measures with symmetries* — Let us tackle the task of determining all Möbius fibred valuations  $F = (f_\gamma)_{\gamma \in \mathcal{C}}$ , if any, with the following symmetry properties:

$$f_1(a) = f_1(c), \quad f_{a \cdot c}(a) = f_{a \cdot c}(c), \quad (27)$$

and such that  $f_\gamma(x) > 0$  whenever  $x$  is a letter enabled at  $\gamma$ .

4.4.4 *Parameters* — In a direct approach intending to illustrate the above notions, let us introduce the following parameters that entirely encode our problem:

$$\begin{aligned} p &= f_1(a) = f_1(c) & q &= f_1(b) \\ r &= f_a(c) & r' &= f_a(a) \\ t &= f_c(a) & t' &= f_c(c) \\ u &= f_{a \cdot c}(a) = f_{a \cdot c}(c) & q' &= f_b(b) \end{aligned}$$

We write down the concurrency equations (Proposition 4.2):

$$\begin{array}{lll}
\text{At } 1: & f_1(a)f_a(c) = f_1(c)f_c(a) & \text{i.e. } pr = pt \\
\text{At } a: & f_a(a)f_1(c) = f_a(c)f_{a \cdot c}(a) & \text{i.e. } r'p = ru \\
\text{At } b: & \text{none} & \\
\text{At } c: & f_c(c)f_1(a) = f_c(a)f_{a \cdot c}(c) & \text{i.e. } t'p = tu \\
\text{At } a \cdot c: & f_{a \cdot c}(a)f_c(c) = f_{a \cdot c}(c)f_a(a) & \text{i.e. } ut' = ur'
\end{array}$$

Since  $u, p > 0$  we deduce at once:  $r = t$  and  $r' = t'$  and we are thus left with parameters  $p, q, q', r, r', u$  with the only equation:

$$r'p = ru. \quad (28)$$

We write down the Möbius equations  $h_\gamma(1) = 0$  for  $\gamma$  ranging over  $\mathcal{C}$ :

$$\text{At } 1: \quad 1 - 2p - q + pr = 0 \quad (29)$$

$$\text{At } a: \quad 1 - r' - r + r'p = 0 \quad (30)$$

$$\text{At } b: \quad 1 - q' = 0 \quad (31)$$

$$\text{At } c: \quad \text{same as (30)}$$

$$\text{At } a \cdot c: \quad 1 - 2u + ur' = 0 \quad (32)$$

4.4.5 *Solutions* — Equation (32) re-writes as:  $u(1 - r') = 1 - u$ . Replacing  $r'p$  by  $ru$  in (30), thanks to (28), and then multiplying by  $u$  yields thus:

$$(1 - u)(1 - ru) = 0.$$

Assume that  $u \neq 1$ . Then  $ru = 1$ , but since both  $r$  and  $u$  lie in  $(0, 1]$  it implies  $r = u = 1$ , and similarly thanks to (28),  $r' = p = 1$ , but then  $q = 0$  by (29), contradicting our assumption  $f_\gamma(x) > 0$  for all enabled letters  $x$ .

Therefore  $u = 1$ , and thus  $r' = 1$  by (32), and  $p = r$  by (28). We now only have two parameters  $p$  and  $q$  related by  $1 - 2p - q + p^2 = 0$ , yielding  $p = 1 - \sqrt{q}$ . Here the additional inequalities from Möbius conditions are trivially satisfied for  $q \in (0, 1)$ .

We obtain thus a continuum of Markov measures satisfying the symmetry conditions (27). All the parameters of the graded valuation corresponding to the Markov measure are deduced from  $q$ , using the above relations. They are graphically gathered in Figure 6-(b). So for instance, starting from the tiling correspond to the empty clique, the probability of obtaining the trace  $a^2 \cdot c^2 \cdot b^2 \cdot a$  is:

$$\begin{aligned}
\mathbb{P}_1(\uparrow a^2 \cdot c^2 \cdot b^2 \cdot a) &= f_1(a)f_a(a)f_1(c)f_c(c)f_1(b)f_b(b)f_1(a) \\
&= pr'pr'qq'p = p^3q = (1 - \sqrt{q})^3q.
\end{aligned} \quad (33)$$

It could have been computed alternatively, yielding the same result:

$$\begin{aligned}
\mathbb{P}_1(\uparrow a^2 \cdot c^2 \cdot b^2 \cdot a) &= \mathbb{P}(\uparrow a \cdot c \cdot c \cdot a \cdot b^2 \cdot a) \\
&= f_1(a)f_a(c)f_{ac}(c)f_a(a)f_1(b)f_b(b)f_1(a) \\
&= prur'qq'p = p^3q = (1 - \sqrt{q})^3q.
\end{aligned} \quad (34)$$

In order to construct Markov measures for this example, we have used a direct approach, writing down the concurrency equations and the Möbius equations, and solving them by hand. We shall see below in next section a more elegant way of constructing Markov measures.

## 5—Uniform measures

In the previous section, we have characterised Markov measures for total actions by means of a finite number of probabilistic parameters. In § 4.4, we have seen examples of Markov measures  $(\mathbb{P}_\alpha)_{\alpha \in X}$  for a *partial* action  $X \times \mathcal{M} \rightarrow X$ , with the following additional property:

$$\forall \alpha \in X \quad \forall x \in \mathcal{M} \quad \mathbb{P}_\alpha(\uparrow x) > 0 \iff x \in \mathcal{M}_\alpha, \quad (35)$$

where  $\mathcal{M}_\alpha = \{x \in \mathcal{M} : \alpha \cdot x \neq \perp\}$  is the trace language associated with the partial action (§ 3.2.4).

Property (35) is natural when dealing with a partial action: the probabilistic dynamics should only concern enabled actions.

It is not obvious, in view of the results from the previous section, that Markov measures satisfying (35) necessarily exist for every partial action. It is the aim of this section to prove this result for irreducible actions. The measure we obtain generalises to trace monoids the uniform measure for irreducible sub-shifts of finite type, due to Parry [22, 20, 16].

### 5.1 — Growth series and characteristic root

We put aside for a moment the probabilistic dynamics, and focus on the combinatorics of partial actions of trace monoids. In subsection 5.3, we will see how both aspects, probability and combinatorics, combine together and lead us to a notion of uniform measure.

We consider a partial action  $X \times \mathcal{M} \rightarrow X$  of a trace monoid over a finite set  $X$ .

5.1.1 *Growth series* — For each  $\alpha \in X \setminus \{\perp\}$ , let  $Z_\alpha(t)$  be the power series defined by:

$$Z_\alpha(t) = \sum_{x \in \mathcal{M}_\alpha} t^{|x|},$$

where  $\mathcal{M}_\alpha$  denotes the trace language associated with the action. Let  $t_\alpha$  be the radius of convergence of  $Z_\alpha$ . Since  $Z_\alpha$  has only non negative terms,  $t_\alpha$  is a singularity of  $Z_\alpha$ .

For each integer  $k \geq 0$ , let  $\mathcal{M}_\alpha(k)$  denote the subset of traces  $\mathcal{M}_\alpha(k) = \{x \in \mathcal{M}_\alpha : |x| = k\}$ . Since at least one letter is enabled at each state, it is easy to see that  $\#\mathcal{M}_\alpha(k) \geq 1$ . And since  $Z_\alpha(t)$  also writes as  $Z_\alpha(t) = \sum_{k \geq 0} \#\mathcal{M}_\alpha(k)t^k$ , we have  $t_\alpha \leq 1$ .

5.1.2 *Characteristic root of an irreducible action* — We show below that the real  $t_\alpha$ , which characterises the growth of the monoid acting on  $X$  and starting from  $\alpha$ , does not depend on  $\alpha$  if the action is irreducible.

• **Lemma 5.1**—If  $\alpha$  leads to  $\beta$  ( $\alpha \Rightarrow \beta$  with the notation of § 3.4.1), then  $t_\alpha \leq t_\beta$ .

*Proof.* Recall that we are only dealing with power series with non negative coefficients. Assuming that  $\alpha \Rightarrow \beta$  holds, let  $x \in \mathcal{M}_\alpha$  be such that  $\beta = \alpha \cdot x$ . Then we have  $x \cdot y \in \mathcal{M}_\alpha$  for every  $y \in \mathcal{M}_\beta$ , and therefore:

$$Z_\alpha(t) \geq \sum_{y \in \mathcal{M}_\beta} t^{|x \cdot y|} \geq t^{|x|} Z_\beta(t).$$

It follows that  $Z_\alpha(t) < \infty \implies Z_\beta(t) < \infty$ , whence the inequality  $t_\beta \geq t_\alpha$ .  $\square$

• **Corollary 5.2**—If the partial action is irreducible, then  $t_\alpha$  is independent of  $\alpha$ .



*Proof.* Any two states  $\alpha$  and  $\beta$  of an irreducible partial action communicate:  $\alpha \Rightarrow \beta$  and  $\beta \Rightarrow \alpha$ . Hence  $t_\alpha = t_\beta$  by Lemma 5.1.  $\square$

The corollary justifies the following definition.

• **Definition 5.3**—*For an irreducible partial action, the common value  $t_\alpha$ , for  $\alpha$  ranging over  $X$ , is called the characteristic root of the action. We denote it by  $t_0$ .*

We shall now study an inversion formula, which extends to actions of trace monoids the first Möbius inversion formula valid for trace monoids (§ 2.2.2). It will justify the term of “root” for  $t_0$ .

## 5.2 — An inversion formula for actions of trace monoids

5.2.1 *Formal fibred series over non commutative variables* — We extend the notion of formal series over partially commutative variables [26, 9] in order to take into account not only the combinatorics of a trace monoid, but also the combinatorics of a trace monoid acting on a finite set.

• **Definition 5.4**—*Given the partial action  $X \times \mathcal{M} \rightarrow X$ , we consider for each pair  $(\alpha, \beta)$  of states, the trace language  $\mathcal{M}_{\alpha, \beta}$  of those traces leading from  $\alpha$  to  $\beta$ , hence defined by:*

$$\mathcal{M}_{\alpha, \beta} = \{x \in \mathcal{M} : x \in \mathcal{M}_\alpha \wedge \alpha \cdot x = \beta\}. \quad (36)$$

*A fibred formal series is a square matrix  $F = (F_{\alpha, \beta})_{(\alpha, \beta) \in X \times X}$ , where each  $F_{\alpha, \beta}$  is a mapping  $F_{\alpha, \beta} : \mathcal{M} \rightarrow \mathbb{Z}$ , such that holds:*

$$\forall x \in \mathcal{M} \quad F_{\alpha, \beta}(x) \neq 0 \implies x \in \mathcal{M}_{\alpha, \beta}.$$

*We see  $F_{\alpha, \beta}$  as a formal sum indexed by  $\mathcal{M}_{\alpha, \beta}$ . We denote by  $\mathbb{Z}\langle\langle \mathcal{M}, X \rangle\rangle$  the set of fibred formal series.*

*An element  $F \in \mathbb{Z}\langle\langle \mathcal{M}, X \rangle\rangle$  is a formal fibred polynomial whenever  $F_{\alpha, \beta}$  has a finite support, for any  $(\alpha, \beta) \in X \times X$ .*

5.2.2  *$\mathbb{Z}$ -algebra structure for  $\mathbb{Z}\langle\langle \mathcal{M}, X \rangle\rangle$*  — If  $F_{\alpha, \beta}$  and  $F'_{\alpha, \beta}$  are two formal sums indexed by  $\mathcal{M}_{\alpha, \beta}$ , we define their sum  $F_{\alpha, \beta} + F'_{\alpha, \beta}$  term by term, leading to another formal sum indexed by  $\mathcal{M}_{\alpha, \beta}$ . The scalar multiplication  $\lambda F_{\alpha, \beta}$  is defined by the scalar multiplication of each term.

If  $F_{\alpha, \beta}$  and  $G_{\beta, \gamma}$  are two formal sums indexed by  $\mathcal{M}_{\alpha, \beta}$  and  $\mathcal{M}_{\beta, \gamma}$  respectively, we can define their Cauchy product  $F_{\alpha, \beta} G_{\beta, \gamma}$  as the a formal sum indexed by  $\mathcal{M}_{\alpha, \gamma}$ , each term being defined by the following finite sum:

$$\forall x \in \mathcal{M}_{\alpha, \gamma} \quad (F_{\alpha, \beta} G_{\beta, \gamma})(x) = \sum_{\substack{(y, z) \in \mathcal{M}_{\alpha, \beta} \times \mathcal{M}_{\beta, \gamma} : \\ y \cdot z = x}} F_{\alpha, \beta}(y) G_{\beta, \gamma}(z) \quad (37)$$

Next, we define the product of two fibred formal series  $F$  and  $G$  by the matrix product:

$$(FG)_{\alpha, \gamma} = \sum_{\beta \in X} F_{\alpha, \beta} G_{\beta, \gamma},$$

where each product in the sum is taken as in (37).

Then clearly,  $\mathbb{Z}\langle\langle \mathcal{M}, X \rangle\rangle$  has the structure of a  $\mathbb{Z}$ -algebra, of which fibred formal polynomials are a sub-algebra. The identity element is the identity matrix  $I = (I_{\alpha, \beta})_{(\alpha, \beta) \in X \times X}$ , which terms are defined by:

$$\forall (\alpha, \beta) \in X \times X \quad \forall x \in \mathcal{M}_{\alpha, \beta} \quad I_{\alpha, \beta}(x) = \mathbf{1}_{\{\alpha = \beta\}} \mathbf{1}_{\{x = 1\}}.$$

5.2.3 *Zeta series and inversion formula* — We shall be interested by two elements of  $\mathbb{Z}\langle\langle\mathcal{M}, X\rangle\rangle$  in particular.

• **Definition 5.5**—*The zeta fibred formal series is the element  $\zeta \in \mathbb{Z}\langle\langle\mathcal{M}, X\rangle\rangle$  defined by:*

$$\forall(\alpha, \beta) \in X \times X \quad \forall x \in \mathcal{M}_{\alpha, \beta} \quad \zeta_{\alpha, \beta}(x) = 1.$$

For each pair  $(\alpha, \beta) \in X \times X$ , let  $\mathcal{C}_{\alpha, \beta}$  denote the set of cliques leading from  $\alpha$  to  $\beta$ , if any:  $\mathcal{C}_{\alpha, \beta} = \mathcal{C} \cap \mathcal{M}_{\alpha, \beta}$ . The Möbius fibred polynomial is the element  $\mu$  defined by:

$$\forall(\alpha, \beta) \in X \times X \quad \forall x \in \mathcal{M}_{\alpha, \beta} \quad \mu_{\alpha, \beta}(x) = (-1)^{|x|} \mathbf{1}_{\{x \in \mathcal{C}_{\alpha, \beta}\}}.$$

Then we have the following inversion formula.

• **Theorem 5.6**—*The Möbius fibred polynomial is the formal inverse of the zeta fibred formal series:*

$$\mu\zeta = \zeta\mu = I$$

*Proof.* By definition of the product in  $\mathbb{Z}\langle\langle\mathcal{M}, X\rangle\rangle$ , we have for  $(\alpha, \beta) \in X \times X$  and for  $x \in \mathcal{M}_{\alpha, \beta}$ :

$$\begin{aligned} (\mu\zeta)_{\alpha, \beta}(x) &= \sum_{\gamma \in X} \left( \sum_{y \in \mathcal{C}_{\alpha, \gamma} : y \leq x} (-1)^{|y|} \right) \\ &= \sum_{y \in \mathcal{C}_{\alpha} : y \leq x} (-1)^{|y|}, \end{aligned}$$

where  $\mathcal{C}_{\alpha}$  denotes  $\mathcal{C}_{\alpha} = \mathcal{C} \cap \mathcal{M}_{\alpha}$ . The binomial formula yields thus:

$$(\mu\zeta)_{\alpha, \beta}(x) = \begin{cases} 1, & \text{if } x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

But  $x = 1$  implies  $\alpha = \beta$ , and so we have proved  $\mu\zeta = I$ . The equality  $\zeta\mu = I$  is proved similarly.  $\square$

5.2.4 *A morphism toward a single formal variable* — In order to exploit the above inversion formula (Theorem 5.6) for counting purposes, we follow a classical pattern by projecting fibred formal series onto series over a single formal variable.

• **Definition 5.7**—*The algebra of single fibred formal series is the algebra  $\mathbb{Z}_X[[t]]$  of square matrices indexed by  $X$ , where each entry is a formal series over the single formal variable  $t$ .*

*The algebra  $\mathbb{Z}_X[t]$  of single formal polynomials is the sub-algebra of  $\mathbb{Z}_X[[t]]$  such that each entry is a polynomial in  $t$ .*

It must be noted that if  $F$  and  $G$  are two formal series over  $t$  given under the form:

$$F = \sum_{x \in \mathcal{M}} F_x t^{|x|}, \quad G = \sum_{x \in \mathcal{M}} G_x t^{|x|},$$

then their product is given by the following Cauchy product formula with respect to the multiplication in the monoid  $\mathcal{M}$ :

$$FG = \sum_{x \in \mathcal{M}} \left( \sum_{(y, z) \in \mathcal{M} \times \mathcal{M} : y \cdot z = x} F_y G_z \right) t^{|x|}. \quad (38)$$

- **Proposition 5.8**—The mapping  $\Phi : \mathbb{Z}\langle\langle\mathcal{M}, X\rangle\rangle \rightarrow \mathbb{Z}_X[[t]]$  defined as follows, for  $F \in \mathbb{Z}\langle\langle\mathcal{M}, X\rangle\rangle$ :

$$\forall(\alpha, \beta) \in X \times X \quad (\Phi F)_{\alpha, \beta} = \sum_{x \in \mathcal{M}} F_{\alpha, \beta}(x) t^{|x|}$$

is a morphism of  $\mathbb{Z}$ -algebra.

*Proof.* The property  $\Phi I = I$  is obvious. The property  $\Phi(FG) = \Phi(F)\Phi(G)$  is the matter of a simple verification left to the reader based on (38).  $\square$

**5.2.5 The theta polynomial** — We now introduce the  $Z$  series, which strong combinatorial meaning justifies the introduction of the above material.

- **Definition 5.9**—The  $Z$  series is the single fibred formal series defined by:

$$\forall(\alpha, \beta) \in X \times X \quad Z_{\alpha, \beta}(t) = \sum_{x \in \mathcal{M}_{\alpha, \beta}} t^{|x|}$$

The Möbius single polynomial  $\mu \in \mathbb{Z}_X[t]$  is defined by:

$$\forall(\alpha, \beta) \in X \times X \quad \mu_{\alpha, \beta}(t) = \sum_{\gamma \in \mathcal{C}_{\alpha, \beta}} (-1)^{|\gamma|} t^{|\gamma|}.$$

The theta polynomial is the standard polynomial  $\theta(t) \in \mathbb{Z}[t]$  defined as the following determinant:

$$\theta(t) = \det \mu(t).$$

For instance, if the partial action is that of a sub-shift of finite type generated by an incidence matrix  $M$  over the set of states, then the single Möbius polynomial is  $\mu(t) = I - tM$ . See computations of  $\mu(t)$  and of  $\theta(t)$  for actions with asynchrony in § 5.4.2.

- **Theorem 5.10**—The Möbius single polynomial is the formal inverse in  $\mathbb{Z}_X[[t]]$  of the  $Z$  series:

$$\mu(t)Z(t) = Z(t)\mu(t) = I,$$

where  $I$  is the identity matrix defined by  $I_{\alpha, \beta}(t) = \mathbf{1}_{\{\alpha = \beta\}}$ .

*Proof.* This derives by applying the morphism  $\Phi$  of Proposition 5.8 to the inversion formula stated in Theorem 5.6, since  $\Phi(\zeta) = Z(t)$  and  $\Phi(\mu) = \mu(t)$ .  $\square$

- **Theorem 5.11**—The characteristic root  $t_0$  of the action  $X \times \mathcal{M} \rightarrow X$  is the smallest positive root of the theta polynomial.

*Proof.* Let  $t \in (0, t_0)$ . Then all terms in  $Z(t)$  are non negative convergent series, therefore the equality  $\mu(t)Z(t) = I$  holds in the space of real matrices  $M_n(\mathbb{R})$ , with  $n = |X|$ . Therefore  $\theta(t) \neq 0$ .

Assume, seeking a contradiction, that  $\theta(t_0) \neq 0$ . Then  $\mu(t)$  is invertible for all  $t$  in the closed interval  $[0, t_0]$ . Therefore the real matrices  $Z(t)$  are uniformly bounded on  $[0, t_0]$ .

Observe now that, for  $t < t_0$  and for  $\alpha \in X$ , the line indexed by  $\alpha$  of the real matrix  $Z(t)$  has its terms that sum up to the convergent growth series  $Z_\alpha(t)$ . Hence we obtain that  $Z_\alpha(t)$  is bounded on  $[0, t_0]$ , a contradiction which completes the proof of the theorem.  $\square$

5.2.6 *Reduction to the standard Möbius inversion formula* — Theorem 5.10 entails the first Möbius inversion formula (§ 2.2.2) as a particular case. Consider indeed the trivial action of a trace monoid  $\mathcal{M}$  over a singleton  $\{*\}$ . Then  $Z(t)$  identifies with the growth series  $H(t) = \sum_{x \in \mathcal{M}} t^{|x|}$ , the Möbius fibred polynomial is simply the Möbius polynomial  $\mu_{\mathcal{M}}(t) = \sum_{c \in \mathcal{C}} (-1)^{|c|} t^{|c|}$  and the formula in Theorem 5.6 is the inversion formula (1).

### 5.3 — Construction of the uniform measure

5.3.1 *Measures on  $\overline{\mathcal{M}}$*  — We recall that  $\overline{\mathcal{M}} = \mathcal{M} \cup \partial\mathcal{M}$  denotes the set of generalised traces. It is equipped with a metrisable topology, for which it is a compact space (§ 2.3.5).

Assume that the partial action  $X \times \mathcal{M} \rightarrow X$  is irreducible. For  $\alpha \in X$  and for  $t \in (0, t_0)$ , let  $\nu_{\alpha,t}$  denote the probability measure on  $\overline{\mathcal{M}}$  defined by:

$$\nu_{\alpha,t} = \frac{1}{Z_{\alpha}(t)} \sum_{x \in \mathcal{M}_{\alpha}} t^{|x|} \delta_{\{x\}}, \quad Z_{\alpha}(t) = \sum_{x \in \mathcal{M}_{\alpha}} t^{|x|}.$$

• **Lemma 5.12**—*For each state  $\alpha \in X$  and for each trace  $x \in \mathcal{M}_{\alpha}$ , the full elementary cylinder  $\uparrow x$  defined by:*

$$\uparrow x = \{\xi \in \overline{\mathcal{M}} : x \leq \xi\}$$

*is given the  $\nu_{\alpha,t}$ -probability:*

$$\nu_{\alpha,t}(\uparrow x) = t^{|x|} \frac{Z_{\alpha \cdot x}(t)}{Z_{\alpha}(t)}.$$

*Proof.* Since  $\nu_{\alpha,t}$  only charges  $\mathcal{M}$ ,  $\nu_{\alpha,t}(\uparrow x)$  is given by the following sum:

$$\nu_{\alpha,t}(\uparrow x) = \frac{1}{Z_{\alpha}(t)} \sum_{y \in \mathcal{M}_{\alpha} : x \leq y} t^{|y|}.$$

The traces  $y \in \mathcal{M}_{\alpha}$  such that  $x \leq y$  are of the form  $y = x \cdot z$  with  $z \in \mathcal{M}_{\alpha \cdot x}$ , hence:

$$\nu_{\alpha,t}(\uparrow x) = \frac{1}{Z_{\alpha}(t)} \sum_{z \in \mathcal{M}_{\alpha \cdot x}} t^{|x \cdot z|} = t^{|x|} \frac{Z_{\alpha \cdot x}(t)}{Z_{\alpha}(t)}.$$

The proof is complete. □

• **Lemma 5.13**—*For an irreducible partial action, the probability measures  $(\nu_{\alpha,t})_{t < t_0}$  have a weak limit when  $t \rightarrow t_0^-$ . This limit, denoted by  $\nu_{\alpha}$ , only charges  $\partial\mathcal{M}$ , and  $\nu_{\alpha}(\uparrow x) > 0$  for every  $x \in \mathcal{M}_{\alpha}$ .*

*Proof.* First, we fix  $\alpha \in X$ . Since  $\overline{\mathcal{M}}$  is compact, there exists a probability measure  $\nu_{\alpha}$  on  $\overline{\mathcal{M}}$  which is a weak limit of a sequence  $(\nu_{\alpha,t_n})_{n \geq 0}$ , for  $t_n \rightarrow t_0^-$ .

For  $x \in \mathcal{M}$ , the singleton  $\{x\}$  is both open and closed in  $\overline{\mathcal{M}}$ . It has thus an empty topological boundary. Therefore, by the Portemanteau theorem [7], the following equality holds:

$$\nu_{\alpha}(\{x\}) = \lim_{n \rightarrow \infty} \nu_{\alpha,t_n}(\{x\}) = \lim_{n \rightarrow \infty} \frac{t_n^{|x|}}{Z_{\alpha}(t_n)} = 0$$

since  $t_n \leq t_0 \leq 1$  on the one hand, and since  $\lim_{t \rightarrow t_0} Z_{\alpha}(t) = \infty$  on the other hand. The set  $\mathcal{M}$  is countable, and thus  $\nu_{\alpha}(\mathcal{M}) = 0$ .

The elementary cylinder  $\uparrow x$  is also both open and closed in  $\overline{\mathcal{M}}$ . Using again the Portemanteau theorem, we have thus:

$$\nu_\alpha(\uparrow x) = \lim_{n \rightarrow \infty} \nu_{\alpha, t_n}(\uparrow x).$$

Since  $\nu_\alpha(\mathcal{M}) = 0$ , it follows:

$$\nu_\alpha(\uparrow x) = \lim_{n \rightarrow \infty} \nu_{\alpha, t_n}(\uparrow x) = t_0^{|x|} \lim_{n \rightarrow \infty} \frac{Z_{\alpha \cdot x}(t_n)}{Z_\alpha(t_n)},$$

the last equality by virtue of Lemma 5.12.

Focus now on the ratio  $Z_{\alpha \cdot x}(t)/Z_\alpha(t)$ . It is a ratio of power series with non negative coefficients. It has therefore a limit, for  $t \rightarrow t_0^-$ , which is either zero, or a positive real, or  $+\infty$ . We have just seen that, for some sequence  $t_n \rightarrow t_0^-$ , the ratios  $Z_{\alpha \cdot x}(t_n)/Z_\alpha(t_n)$  have a non negative limit. We deduce:

$$\lim_{t \rightarrow t_0^-} \frac{Z_{\alpha \cdot x}(t)}{Z_\alpha(t)} \in [0, \infty) \quad (39)$$

It also follows that the probability measure  $\nu_\alpha$  has its values on elementary cylinders  $\uparrow x$  entirely determined by these limits, independently of the sequence  $(t_n)_n$ . Therefore the family  $(\nu_{\alpha, t})_{t < t_0}$  has  $\nu_\alpha$  as weak limit when  $t \rightarrow t_0^-$ .

It remains only to show  $\nu_\alpha(\uparrow x) > 0$  for  $x \in \mathcal{M}_\alpha$ . It amounts to showing that the limit in (39) is positive. Let  $\beta = \alpha \cdot x$ . Since the action is irreducible, there exists  $y \in \mathcal{M}_\beta$  such that  $\alpha = \beta \cdot y$ . Applying (39) to  $\beta$  and  $y$  in place of  $\alpha$  and  $x$ , we have:

$$\lim_{t \rightarrow t_0^-} \frac{Z_{\beta \cdot y}(t)}{Z_\beta(t)} < \infty$$

which also writes as:

$$\lim_{t \rightarrow t_0^-} \frac{Z_{\alpha \cdot x}(t)}{Z_\alpha(t)} > 0. \quad (40)$$

This completes the proof of the lemma.  $\square$

5.3.2 *The Parry cocycle* — The above lemma has the following consequence.

• **Corollary 5.14**—*For each pair of states  $(\alpha, \beta) \in X \times X$  of an irreducible partial action, the following limit:*

$$\Gamma(\alpha, \beta) = \lim_{t \rightarrow t_0^-} \frac{Z_\beta(t)}{Z_\alpha(t)} \quad (41)$$

*exists and lies in  $(0, \infty)$ .*

*Proof.* Since the action is irreducible, every  $\beta \in X$  writes as  $\beta = \alpha \cdot x$  for some  $x \in \mathcal{M}_\alpha$ . The statement of the corollary was then shown in the course of the proof of Lemma 5.13, in (39) and in (40).  $\square$

• **Definition 5.15**—*For an irreducible partial action  $X \times \mathcal{M} \rightarrow X$ , the Parry cocycle is the function  $\Gamma(\cdot, \cdot) : X \times X \rightarrow (0, \infty)$  defined by (41).*

The term “cocycle” is appropriate, since the definition (41) of  $\Gamma(\cdot, \cdot)$  makes obvious the two following properties:

$$\forall \alpha \in X \quad \Gamma(\alpha, \alpha) = 1 \quad (42)$$

$$\forall (\alpha, \beta, \gamma) \in X \times X \times X \quad \Gamma(\alpha, \gamma) = \Gamma(\alpha, \beta)\Gamma(\beta, \gamma) \quad (43)$$

5.3.3 *Uniform measure* — We can now combine the results obtained so far to obtain the following theorem.

• **Theorem 5.16**—*Let  $X \times \mathcal{M} \rightarrow X$  be an irreducible partial action. Then there exists a Markov measure, denoted  $\nu = (\nu_\alpha)_{\alpha \in X}$  and called the uniform measure, satisfying the following property:*

$$\forall \alpha \in X \quad \forall x \in \mathcal{M}_\alpha \quad \nu_\alpha(\uparrow x) = t_0^{|x|} \Gamma(x, \alpha \cdot x), \quad (44)$$

and  $\nu_\alpha(\uparrow x) = 0$  if  $x \notin \mathcal{M}_\alpha$ . In particular, the support of  $\nu$  coincides with the support of the partial action.

*Proof.* According to Lemma 5.13, we shall define  $\nu_\alpha$ , for each  $\alpha \in X$ , as the weak limit of  $\nu_{\alpha,t}$  for  $t \rightarrow t_0^-$ , and then put  $\nu = (\nu_\alpha)_{\alpha \in X}$ .

The relation (44) follows from an application of the Portemanteau theorem as in the proof of Lemma 5.13, starting from the result of Lemma 5.12 and taking the limit  $t \rightarrow t_0^-$ . Since the Parry cocycle is positive,  $\nu_\alpha(\uparrow x) > 0$  for all  $x \in \mathcal{M}_\alpha$ .

If  $x \notin \mathcal{M}_\alpha$ , then  $\nu_{\alpha,t}(\uparrow x) = 0$  for all  $t < t_0$  by the very definition of  $\nu_{\alpha,t}$  and since  $\mathcal{M}_\alpha$  is downward closed according to Proposition 3.6. Therefore, passing to the limit still legitimated by the Portemanteau theorem, we obtain  $\nu_\alpha(\uparrow x) = 0$ . Hence  $\nu = (\nu_\alpha)_{\alpha \in X}$  has same support as the action.

It remains to show that  $\nu$  thus defined is Markovian. For this, we verify the validity of the defining relation:

$$\forall \alpha \in X \quad \forall x, y \in \mathcal{M} \quad \nu_\alpha(\uparrow (x \cdot y)) = \nu_\alpha(\uparrow x) \nu_{\alpha \cdot x}(\uparrow y). \quad (45)$$

If  $x \notin \mathcal{M}_\alpha$  or if  $y \notin \mathcal{M}_{\alpha \cdot x}$ , then both members of the above equality are zero. And for  $x \in \mathcal{M}_\alpha$  and  $y \in \mathcal{M}_{\alpha \cdot x}$ , using the cocycle property (43), we have:

$$\begin{aligned} \nu_\alpha(\uparrow (x \cdot y)) &= t_0^{|x \cdot y|} \Gamma(\alpha, \alpha \cdot x \cdot y) \\ &= t_0^{|x|} \Gamma(\alpha, \alpha \cdot x) t_0^{|y|} \Gamma(\alpha \cdot x, (\alpha \cdot x) \cdot y) \\ &= \nu_\alpha(\uparrow x) \nu_{\alpha \cdot x}(\uparrow y) \end{aligned}$$

The proof is complete. □

## 5.4 — Example: uniform measure of a Rabati tiling

We illustrate the construction of the uniform measure on the simple example already encountered in § 4.4 of the 4-Rabati tiling in line.

5.4.1 *Continuing the direct approach* — Let us tackle the problem of determining the uniform measure directly through its Möbius fibred valuation  $F = (f_\gamma)_{\gamma \in \mathcal{C}}$ .

For symmetry reasons, it is clear that  $F$  satisfies the conditions  $f_1(a) = f_1(c)$  and  $f_{a \cdot c}(a) = f_{a \cdot c}(c)$ , hence it fits within the framework previously studied in § 4.4.

At the end of § 4.4.5, we were left with the only parameter  $q = f_1(b)$ , from which all other values characterising the Möbius valuation  $F$  can be deduced. Given the special form of the uniform measure, we have on the one hand:

$$\begin{aligned} f_1(b) &= t_0 \Gamma(1, b) \\ f_b(b) &= t_0 \Gamma(b, 1) \end{aligned}$$

and on the other hand, by the cocycle property (43) and since  $f_b(b) = 1$ :

$$\Gamma(b, 1) = t_0^{-1} \quad \Gamma(1, b) = \Gamma(b, 1)^{-1}$$

Therefore:

$$q = t_0 \Gamma(1, b) = t_0^2$$

Applying the same reasoning for  $p = f_1(a)$  we have, using that  $f_a(a) = r' = 1$ :

$$p = t_0 \Gamma(1, a) \quad 1 = t_0 \Gamma(a, 1) = t_0 \Gamma(1, a)^{-1} = \frac{t_0^2}{p}$$

Therefore  $p = t_0^2 = q$ . But we also found  $1 - 2p - q + p^2 = 0$ , therefore:

$$1 - 3p + p^2 = 0$$

and we recognise that  $p = \frac{3-\sqrt{5}}{2}$  is the only root in  $(0, 1)$  of the Möbius polynomial of the monoid  $\mathcal{M} = \langle a, b, c \mid ac = ca \rangle$ . The characteristic root of the action is:

$$t_0 = \sqrt{\frac{3-\sqrt{5}}{2}}.$$

Since we have obtained the value of  $t_0$  and we can determine the Möbius fibred valuation, we can deduce the values of the Parry cocycle, through the formula  $f_\gamma(x) = t_0^{|\gamma|} \Gamma(\gamma, \varphi(\gamma, x))$ . By the cocycle property, it is enough to give the values of  $\Gamma(1, \cdot)$ :

$$\Gamma(1, a) = t_0 \quad \Gamma(1, b) = t_0 \quad \Gamma(1, c) = t_0 \quad \Gamma(1, a \cdot c) = t_0^2$$

We obtain thus the following formula for the cocycle:

$$\forall \gamma, \gamma' \in \mathcal{C} \quad \Gamma(\gamma, \gamma') = t_0^{|\gamma'| - |\gamma|}. \quad (46)$$

We can check that we obtain the same value for  $\mathbb{P}_1(\uparrow x)$ , with  $x = a^2 \cdot c^2 \cdot b^2 \cdot a$ , as we found in (33) and in (34):

$$\mathbb{P}_1(\uparrow x) = t_0^{|\uparrow x|} \Gamma(1, \varphi(1, x)) = t_0^7 \Gamma(1, a) = t_0^8,$$

which corresponds to the value  $p^3 q$  found previously, since  $p = q = t_0^2$ .

**5.4.2 Using the theta polynomial** — We could also have obtained the value of  $t_0$  by applying the result of Theorem 5.11, which implies to determine the theta polynomial of the action. The Möbius single fibred polynomial (Definition 5.9 in § 5.2.5) is the following matrix indexed by  $\mathcal{C}$ :

$$\mu(t) = \begin{matrix} & 1 & a & b & c & a \cdot c \end{matrix} \begin{pmatrix} 1 & -t & -t & -t & t^2 \\ -t & 1 & 0 & t^2 & -t \\ -t & 0 & 1 & 0 & 0 \\ -t & t^2 & 0 & 1 & -t \\ t^2 & -t & 0 & -t & 1 \end{pmatrix}$$

The theta polynomial is the determinant of the above matrix, found to be equal to:

$$\theta(t) = (1 - 3t^2 + t^4)(1 - t^2)^2$$

Via the result of Theorem 5.11, we recover that  $t_0^2$  is the root in  $(0, 1)$  of the Möbius polynomial  $1 - 3p + p^2$  of the monoid.

We shall obtain yet another time this result by studying more generally tip-top actions, of which Rabati tiling actions are a particular case (§ 3.3.3–3.3.5).

## 5.5 — Example: uniform measure of the tip-top action

In this subsection, we study the uniform measure associated with the tip-top action of a trace monoid over its set of cliques (§ 3.3.3).

We recall from § 3.3.5 that the action of flips on a Rabati tiling is a particular case of a tip-top action. Hence our study will allow us to re-obtain results on the uniform measure of the 4-Rabati tiling in line, which we previously obtained by hand through Möbius equations and symmetry considerations in § 4.4 and § 5.4.

Our first result, below, shows that the presence of 1's on all ascending arrows in the graph of Figure 6–(b) is due to a general fact.

• **Lemma 5.17**—*Let  $\mathbb{P} = (\mathbb{P}_\gamma)_{\gamma \in \mathcal{C}}$  be a Markov measure on the tip-top action  $\mathcal{C} \times \mathcal{M} \rightarrow \mathcal{C}$ , and let  $(f_\gamma)_{\gamma \in \mathcal{C}}$  be the associated Möbius valuation. Then holds:*

$$\forall \gamma \in \mathcal{C} \quad \forall a \in \Sigma \quad a \leq \gamma \implies f_\gamma(a) \in \{0, 1\}.$$

*Proof.* Let  $\gamma \in \mathcal{C}$  and let  $a \in \Sigma$  be such that  $a \leq \gamma$ . Then by induction, it is easy to see that for any trace  $x \in \mathcal{M}_\gamma$  of the form  $x = a_1 \cdot \dots \cdot a_n$  with  $a_i \neq a$  for all integers  $i \in \{1, \dots, n\}$ , then  $a_i \parallel a$  holds, and  $a \in \mathcal{M}_{\gamma \cdot x}$  also holds. In words: as long as  $a$  has not been played, it cannot be disabled.

Therefore, assuming that  $f_\gamma(a) > 0$  holds, then the Markov chain of states-and-cliques has positive probability to hit a state-and-clique where the clique involves  $a$ , and the set of such states-and-cliques will remain accessible as long as no clique involving  $a$  has been produced. Hence this will eventually occur with probability 1.

Returning to the valuation point of view, it implies that the cylinder  $\uparrow a$  has  $\mathbb{P}_\gamma$ -probability 1. Hence  $f_\gamma(a) = 1$ , which was to be proved.  $\square$

• **Corollary 5.18**—*Let  $(f_\gamma)_{\gamma \in \mathcal{C}}$  be the Möbius fibred valuation associated with the uniform measure of the tip-top action. Then holds:*

1.  $\forall \gamma \in \mathcal{C} \quad \forall a \in \Sigma \quad a \leq \gamma \implies f_\gamma(a) = 1.$
2.  $\forall \gamma \in \mathcal{C} \quad f_\gamma(\gamma) = 1.$

*Proof.* Let  $\gamma \in \mathcal{C}$  and let  $a \in \Sigma$  be such that  $a \leq \gamma$ . By Theorem 5.16, the uniform measure has same support as the support of the action. Therefore, since  $a \in \mathcal{M}_\gamma$  if  $a \leq \gamma$ , one has  $f_\gamma(a) > 0$  and thus  $f_\gamma(a) = 1$  by Lemma 5.17, which proves point 1.

For point 2, let  $\gamma \in \mathcal{C}$ . Then  $\gamma$  writes as  $\gamma = a_1 \cdot \dots \cdot a_i$ , where the  $a_i$  are pairwise parallel. By the chain rule for the fibred valuation on the one hand, and using the result just proved on the other hand, we have:

$$f_\gamma(\gamma) = f_\gamma(a_1) f_{a_2 \dots a_i}(a_2 \cdot \dots \cdot a_i) = f_{a_2 \dots a_i}(a_2 \cdot \dots \cdot a_i).$$

Hence, by induction,  $f_\gamma(\gamma) = 1$ , which was to be proved.  $\square$

The structure of the uniform measure of the tip-top action is then entirely described by the following result. We re-obtain in particular the values for the characteristic root and for the Parry cocycle found in § 5.4.1 for the 4-Rabati tiling in line, and also in § 5.4.2 by using the theta polynomial for the characteristic root.

• **Theorem 5.19**—*Let  $\mathcal{M}$  be a trace monoid, of Möbius polynomial  $\mu_{\mathcal{M}}$  (§ 2.2.1), and let  $p_0 \in (0, 1)$  be the root of smallest modulus of  $\mu_{\mathcal{M}}$ .*

*Then the characteristic root  $t_0$  of the tip-top action  $\mathcal{C} \times \mathcal{M} \rightarrow \mathcal{C}$  is given by  $t_0 = \sqrt{p_0}$ . The Parry cocycle of the action is given by:*

$$\forall (\gamma, \gamma') \in \mathcal{C} \times \mathcal{C} \quad \Gamma(\gamma, \gamma') = t_0^{|\gamma'| - |\gamma|}. \quad (47)$$



*Proof.* Let  $\nu = (\nu_\alpha)_{\alpha \in \mathcal{C}}$  be the uniform measure of the tip-top action, and let  $(f_\gamma)_{\gamma \in \mathcal{C}}$  be the associated Möbius valuation.

Consider a clique  $\gamma \in \mathcal{C}$ . Then  $\gamma$  is also an action enabled at state  $\gamma$ , the action of which yields to the empty clique 1. Given the result of Corollary 5.18, point 2, on the one hand, and the form (44) of the uniform measure, we have thus:

$$f_\gamma(\gamma) = 1 = t_0^{|\gamma|} \Gamma(\gamma, 1), \quad \Gamma(\gamma, 1) = t_0^{-|\gamma|}.$$

The cocycle relation yields  $\Gamma(1, \gamma) = \Gamma(\gamma, 1)^{-1} = t_0^{|\gamma|}$ . Using again the cocycle relation, we obtain for any two cliques  $\gamma, \gamma' \in \mathcal{C}$ :

$$\Gamma(\gamma, \gamma') = \Gamma(\gamma, 1) \Gamma(1, \gamma') = t_0^{|\gamma'| - |\gamma|},$$

proving (47).

We obtain also the following form for the restriction to  $\mathcal{C}$  of  $f_1(\cdot)$ :

$$\forall \gamma \in \mathcal{C} \quad f_1(\gamma) = t_0^{|\gamma|} \Gamma(1, \gamma) = t_0^{2|\gamma|}.$$

Let  $h_1(\cdot)$  be the Möbius transform of  $f_1(\cdot)$ . Then by the Möbius condition for  $f_1$ , we have  $h_1(1) = 0$  which is equivalent to  $\mu_{\mathcal{M}}(t_0^2) = 0$ , hence  $t_0^2$  is a root of  $\mu_{\mathcal{M}}$ . The Möbius conditions also impose  $h_1(\gamma) \geq 0$  for all non empty cliques  $\gamma \in \mathcal{C}$ . Lemma 5.20 below shows that among the roots of  $\mu_{\mathcal{M}}$ , only the root of smallest modulus satisfies these conditions. This completes the proof.  $\square$

In the course of the above proof, we have used the following lemma. The key ingredient is the uniqueness of uniform measures shown in [4].

• **Lemma 5.20**—*Let  $\mathcal{M}$  be a trace monoid, of Möbius polynomial  $\mu_{\mathcal{M}}(t)$ . Let  $t$  be a non negative real, let  $f : \mathcal{M} \rightarrow \mathbb{R}$  be the uniform valuation defined by  $f(x) = t^{|x|}$ , and let  $h : \mathcal{C} \rightarrow \mathbb{R}$  be the Möbius transform of  $f|_{\mathcal{C}}$ . Assume that  $h(1) = 0$  and that  $h(c) \geq 0$  for all non empty cliques  $c \in \mathcal{C}$ . Then  $t$  is the root of smallest modulus of  $\mu_{\mathcal{M}}$ .*

*Proof.* Let  $D$  be the dependence relation of the monoid  $\mathcal{M} = \mathcal{M}(\Sigma, I)$ , defined by  $D = (\Sigma \times \Sigma) \setminus I$ . The monoid  $\mathcal{M}$  is said to be irreducible—not to be confused with the irreducibility of actions—if  $(\Sigma, D)$  is connected as an unoriented graph.

The proof of the lemma is based on the following fact: there exists a unique Bernoulli measure  $\nu$  on  $\partial\mathcal{M}$  such that  $\nu(\uparrow x) = t^{|x|}$  for all  $x \in \mathcal{M}$  and for some real  $t > 0$ , and  $t$  is the root  $p_0$  of smallest modulus of  $\mu_{\mathcal{M}}$ . This is proved in Theorem 1.6 of [4] if  $\mathcal{M}$  is irreducible. But it is true also if  $\mathcal{M}$  is not irreducible, by considering the projection of  $\nu$  to the component of  $\mathcal{M}$ , of which the smallest root of the Möbius polynomial coincides with  $p_0$  (see [18, Prop. 4.6] and [5, Prop. 1]).

Now, given the real number  $t$  and the valuation  $f$  as in the statement of the lemma, we claim that there exists a Bernoulli measure  $\nu$  on  $\partial\mathcal{M}$  such that  $\nu(\uparrow x) = f(x)$ . Indeed, considering the unique total and trivial action of  $\mathcal{M}$  on the singleton  $\{*\}$ , then the valuation  $f$  can be seen as a fibred Möbius valuation with respect to this action. An application of Theorem 4.8 yields the existence of  $\nu$ . Hence  $t = p_0$ .  $\square$

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